3.6 Permutation Groups

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28. In the dihedral group $D_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that $ba^i = a^{n-i}b$, for all $0 \le i < n$.

    Solution: For $i = 1$, the equation $ba^i = a^{n-i}b$ is just the relation that defines the group. If we assume that the result holds for $i = k$, then for $i = k + 1$ we have

    $$ba^{k+1} = (ba^k)a = (a^{n-k}b)a = a^{n-k}(ba) = a^{n-k}a^{-1}b = a^{n-(k+1)}b\ .$$

    This implies that the result must hold for all i with $0 \le i < n$.

    Comment: This is similar to a proof by induction, but for each given n we only need to worry about a finite number of equations.

29. In the dihedral group $D_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that each element of the form $a^i b$ has order 2.

    Solution: Using the result from Problem 28, we have $(a^i b)^2 = (a^i b)(a^i b) = a^i(ba^i)b = a^i(a^{n-i}b)b = (a^i a^{n-i})(b^2) = a^n e = e$.

30. In S_4, find the subgroup H generated by $(1, 2, 3)$ and $(1, 2)$.

    Solution: Let $a = (1, 2, 3)$ and $b = (1, 2)$. Then H must contain $a^2 = (1, 3, 2)$, $ab = (1, 3)$ and $a^2b = (2, 3)$, and this set of elements is closed under multiplication. (We have just listed the elements of S_3.) Thus $H = \{(1), a, a^2, b, ab, a^2b\} = \{(1),\ (1, 2, 3),\ (1, 3, 2),\ (1, 2),\ (1, 3),\ (2, 3)\}$.

31. For the subgroup H of S_4 defined in the previous problem, find the corresponding subgroup $\sigma H \sigma^{-1}$, for $\sigma = (1, 4)$.

    Comment: Exercise 2.3.13 in the text shows that if $(1, 2, \ldots, k)$ is a cycle of length k and σ is any permutation, then $\sigma(1, 2, \ldots, k)\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$.

    Solution: We need to compute $\sigma\tau\sigma^{-1}$, for each $\tau \in H$. Since $(1, 4)^{-1} = (1, 4)$, we have $(1, 4)(1)(1, 4) = (1)$, and $(1, 4)(1, 2, 3)(1, 4) = (2, 3, 4)$. As a shortcut, we can use Exercise 2.3.13, which shows immediately that $\sigma(1, 2, 3)\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3)) = (4, 2, 3)$. Then we can quickly do the other computations:

    $$\begin{aligned}
    (1,4)(1,3,2)(1,4)^{-1} &= (4,3,2) \\
    (1,4)(1,2)(1,4)^{-1} &= (4,2) \\
    (1,4)(1,3)(1,4)^{-1} &= (4,3) \\
    (1,4)(2,3)(1,4)^{-1} &= (2,3).
    \end{aligned}$$

    Thus $(1, 4)H(1, 4)^{-1} = \{(1),\ (2, 3, 4),\ (2, 4, 3),\ (2, 3),\ (2, 4),\ (3, 4)\}$.

32. Show that each element in A_4 can be written as a product of 3-cycles.

    Solution: We first list the 3-cycles: $(1,2,3)$, $(1,2,4)$, $(1,3,2)$, $(1,3,4)$, $(1,4,2)$, $(1,4,3)$, $(2,3,4)$, and $(2,4,3)$. Rather than starting with each of the other elements and then trying to write them as a product of 3-cycles, it is easier to just look at the possible products of 3-cycles. We have $(1,2,3)(1,2,4) = (1,3)(2,4)$, $(1,2,4)(1,2,3) = (1,4)(2,3)$, $(1,2,3)(2,3,4) = (1,2)(3,4)$, and this accounts for all 12 of the elements in A_4.

33. In the dihedral group $D_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, find the centralizer of a.

    Solution: The centralizer $C(a)$ contains all powers of a, so we have $\langle a \rangle \subseteq C(a)$. This shows that $C(a)$ has at least n elements. On the other hand, $C(a) \neq D_n$, since by definition b does not belong to $C(a)$. Since $\langle a \rangle$ contains exactly half of the elements in D_n, Lagrange's theorem show that there is no subgroup that lies strictly between $\langle a \rangle$ and D_n, so $\langle a \rangle \subseteq C(a) \subseteq D_n$ and $C(a) \neq D_n$ together imply that $C(a) = \langle a \rangle$.

34. Find the centralizer of $(1,2,3)$ in S_3, in S_4, and in A_4.

    Comment: It helps to have some shortcuts when doing the necessary computations. To see that x belongs to $C(a)$, we need to check that $xa = ax$, or, equivalently, that $axa^{-1} = x$. Exercise 2.3.13 provides a quick way to do this in a group of permutations. As noted previously in this section, that exercise shows that if $(1,2,\ldots,k)$ is a cycle of length k and σ is any permutation, then $\sigma(1,2,\ldots,k)\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$.

    Solution: Since any power of an element a commutes with a, the centralizer $C(a)$ always contains the cyclic subgroup $\langle a \rangle$ generated by a. Thus the centralizer of $(1,2,3)$ always contains the subgroup $\{(1), (1,2,3), (1,3,2)\}$.

    In S_3, the centralizer of $(1,2,3)$ is equal to $\langle\, (1,2,3)\, \rangle$, since it is easy to check that $(1,2)$ does not belong to the centralizer, and by Lagrange's theorem a *proper* subgroup of a group with 6 elements can have at most 3 elements.

    To find the centralizer of $(1,2,3)$ in S_4 we have to work a bit harder. Let $a = (1,2,3)$. From the computations in S_3, we know that $(1,2)$, $(1,3)$, and $(2,3)$ do not commute with a. The remaining transpositions in S_4 are $(1,4)$, $(2,4)$, and $(3,4)$. Using Exercise 2.3.13, we have $a(1,4)a^{-1} = (2,4)$, $a(2,4)a^{-1} = (3,4)$, and $a(3,4)a^{-1} = (1,4)$, so no transposition in S_4 commutes with a. For the products of the transpositions, we have $a(1,2)(3,4)a^{-1} = (2,3)(1,4)$, $a(1,3)(2,4)a^{-1} = (2,1)(3,4)$, and $a(1,4)(2,3)a^{-1} = (2,4)(3,1)$, and so no product of transpositions belongs to $C(a)$.

    If we do a similar computation with a 4-cycle, we have $a(x,y,z,4)a^{-1} = (u,v,w,4)$, since a just permutes the numbers x, y, and z. This means that $w \neq z$, so $(u,v,w,4)$ is not equal to $(x,y,z,4)$. Without doing all of the calculations, we can conclude that no 4-cycle belongs to $C(a)$. This accounts for an additional 6 elements. A similar argument shows that no 3-cycle that includes the number 4 as one of its entries can belong to $C(a)$. Since there are 6 elements of this form, we now have a total of 21 elements (out of 24) that are not in $C(a)$, and therefore $C(a) = \langle a \rangle$.

Finally, since A_4 contains the three products of transpositions and the six 3-cycles that include 4, we have nine elements (out of 12 in A_4) that do not commute with $(1,2,3)$. Thus in A_4 we get the same answer: $C(a) = \langle a \rangle$.

35. With the notation of the comments preceding the statement of Theorem 3.6.6, find $\sigma(\Delta_3)$ for each $\sigma \in S_n$.

Solution: By definition, $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.

For $\sigma = (1)$, $\sigma(\Delta_3) = \Delta_3$.

For $\sigma = (1,2,3)$, $\sigma(\Delta_3) = (x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = \Delta_3$.

For $\sigma = (1,3,2)$, $\sigma(\Delta_3) = (x_3 - x_1)(x_3 - x_2)(x_1 - x_2) = \Delta_3$.

For $\sigma = (1,2)$, $\sigma(\Delta_3) = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -\Delta_3$.

For $\sigma = (1,3)$, $\sigma(\Delta_3) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) = -\Delta_3$.

For $\sigma = (2,3)$, $\sigma(\Delta_3) = (x_1 - x_3)(x_1 - x_2)(x_3 - x_2) = -\Delta_3$.

ANSWERS AND HINTS

36. Compute the centralizer of $(1,2)(3,4)$ in S_4.
Answer: $C((1,2)(3,4)) =$
$\{(1),\ (1,2)(3,4),\ (1,2),\ (3,4),\ (1,3)(2,4),\ (1,4)(2,3),\ (1,4,2,3),\ (1,3,2,4)\}$.

37. Show that the group of rigid motions of a cube can be generated by two elements.
Hint: Let ρ be a 90 degree rotation that leaves the top and bottom fixed. Let σ be a 120 degree rotation with axis of rotation the line through two opposite vertices. To help in the computations, you can number the faces of the cube and represent ρ and σ as elements of S_6.

39. Describe the possible shapes of the permutations in A_6. Use a combinatorial argument to determine how many of each type there are.
Answer: 40 have shape (a,b,c); 45 have shape $(a,b)(c,d)$; 144 have shape (a,b,c,d,e); 90 have shape $(a,b,c,d)(e,f)$; 15 have shape $(a,b,c)(d,e,f)$; 1 has shape (a).

40. Find the largest possible order of an element in each of the alternating groups A_5, A_6, A_7, A_8.
Answer: The largest order in A_5 and A_6 is 5. In A_7, the largest possible order is 7. In A_8, the largest possible order is 15.

41. Let G be the dihedral group D_6, denoted by $G = \{a^i b^j | 0 \le i < 6 \text{ and } 0 \le j < 2\}$, where a has order 6, b has order 2, and $ba = a^{-1}b$. Find $C(ab)$.
Answer: $C(ab) = \{e, ab, a^3, a^4 b\}$

44. Is D_{12} isomorphic to $D_4 \times \mathbf{Z}_3$?
Answer: No. *Hint:* Count the number of elements of order 6.