LESSON 10

VECTOR ALGEBRA

1. VECTORS AND SCALARS

All physical quantities are categorized in two parts.

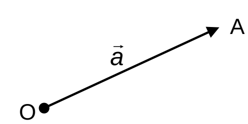
Physical Quantities

Scalar | Vector

Scalar: Quantities which can be specified by magnitude only e.g., mass, density, volume, current, height, distance and speed.

Vector: Quantities which are specified by both magnitude and direction. A vector must obey the vector law of addition e.g., velocity, displacement, acceleration.

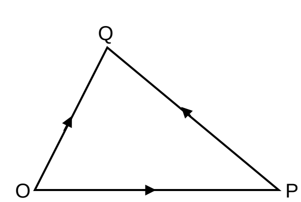
A vector, thus, is geometrically represented by a directed line segment. It is said that $\vec{a}$ is equivalently $\overrightarrow{OA}$ and they are vectorially indistinguishable.



2. POSITION VECTOR OF A POINT

The position vector $\vec{r}$ of any point P with respect to the origin of reference O is a vector $\overrightarrow{OP}$.

For any two points P and Q in the space, the vector $\overrightarrow{PQ}$ can be expressed in terms of their position vectors (p.v.) as



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

Illustration 1

Question: **If $\vec{a}$ and $\vec{b}$ are position vectors of A and B respectively, find the position vector of a point C in AB produced such that $\overrightarrow{AC}$ Z $3\overrightarrow{AB}$.**

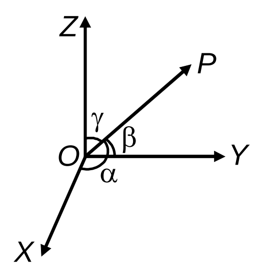
Solution: $\overrightarrow{AC} = 3\overrightarrow{AB} \Rightarrow \vec{c} - \vec{a} = 3\,(\vec{b} - \vec{a})$ where $\vec{c}$ is the position vector of point C

$$\Rightarrow \vec{c} = \vec{a} + 3\vec{b} - 3\vec{a} \quad \Rightarrow \vec{c} = 3\vec{b} - 2\vec{a}.$$

3. DIRECTION COSINES AND DIRECTION RATIOS

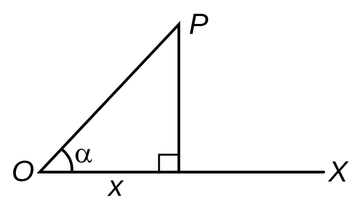
3.1 DIRECTION COSINES

If the position vector of a point P i.e., $\overrightarrow{OP}$ makes angles α, β and γ with the positive direction of x, y and z axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines. They are also denoted by l, m and n respectively.



i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

It can be seen from the figure $\cos\alpha = \dfrac{x}{OP}$



Similarly, $\cos\beta = \dfrac{y}{OP}$ and $\cos\gamma = \dfrac{z}{OP}$

Where OP is the modulus of positive vector of P.

Clearly, $OP = \sqrt{x^2 + y^2 + z^2}$

so, $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{x^2 + y^2 + z^2}{OP^2} = 1$$

$\therefore$ if $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

3.2 DIRECTION RATIOS

If a, b, c three numbers such that $\dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n}$

where l, m, n are direction cosines of a vector $\vec{r}$, then a, b, c are known as direction numbers or direction ratios of $\vec{r}$.

e.g., if $\hat{r} = 2\hat{i} - 3\hat{j} + 10\hat{k}$

then its direction ratios are 2, –3 and 10 or 4, –6 and 20 or any positive multiple of the components or direction cosines of $\vec{r}$.

Two vectors having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

They are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Illustration 2

Question: **A vector $\vec{r}$ has length 21 and direction ratios 2, J3, 6. Find the vector $\vec{r}$.**

Solution: The direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{-3}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{6}{\sqrt{2^2 + (-3)^2 + 6^2}}$$

Since $\vec{r}$ makes an acute angle with x-axis, therefore cos $\alpha > 0$ i.e., $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$\therefore$ $\vec{r} = 21\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$ [using $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$]

or $\vec{r} = 6\hat{i} - 9\hat{j} + 18\hat{k}$

So, components of $\vec{r}$ along *ox*, *oy* and *oz* are $6\hat{i}, -9\hat{j}$ and $18\hat{k}$ respectively.

Illustration 3

Question: **Find the angle between the vectors with direction ratios 4, J3, 5 and 3, 4, 5.**

Solution: Let $\vec{a}$ = a vector parallel to the vector having direction ratios 4, –3, 5 = $4\hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{b}$ = a vector parallel to the vector having direction ratios 3, 4, 5 = $3\hat{i} + 4\hat{j} + 5\hat{k}$.

Let θ be the angle between the given vectors. Then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12 - 12 + 25}{\sqrt{16 + 9 + 25}\sqrt{9 + 16 + 25}} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}.$$

Thus, the angle between the vectors with direction ratios 4, –3, 5 and 3, 4, 5 is 60°.

4. TYPES OF VECTORS

4.1 Zero Vector

A vector whose initial and terminal points coincide, is called a zero vector (or null vector), and denoted as $\vec{0}$. Zero vector can not be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vector $\overrightarrow{AA}, \overrightarrow{BB}$ represent the zero vector.

4.2 Unit Vector

A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. The unit vector in the direction of a given vector $\vec{a}$ is denoted by $\hat{a}$.

4.3 Coinitial Vector

Two or more vectors having the same initial point are called coinitial vectors.

4.4 Collinear Vectors

Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitude and directions.

4.5 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points and written as $\vec{a} = \vec{b}$.

4.6 Negative of a Vector

A vector whose magnitude is the same as that of a given vector (say $\overrightarrow{AB}$), but direction is opposite to that of it, is called negative of the given vector.

For example, vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$ and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Remark:

The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called free vectors. Throughout this chapter, we will be dealing with free vectors only.

5. ADDITION OF VECTORS

The laws, governing the algebra of vectors, are so designed as to be particularly applicable in cases of physical quantities like velocity, acceleration, etc.

Thus two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are added according to the parallelogram law of addition; namely:

If two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are represented in magnitude and direction by two, line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$, their sum $\overrightarrow{c} = \overrightarrow{a} + \overrightarrow{b}$, is represented by the diagonal $\overrightarrow{OC}$ of the completed parallelogram OACB.

Sometimes this is also referred to as the triangle law of addition.

The parallelogram law of addition is

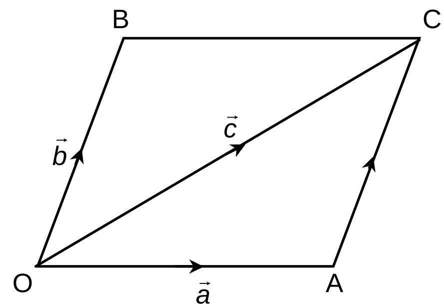
$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

$$(\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c})$$

The triangle law is

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$(\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c})$$



This addition operation, cumulatively, may be had for more than two vectors; and we have

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$$

For addition of more than two vectors we have a polygon laws of vectors addition which is just an extension of triangle law.

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{OF}$$

As a result if terminus of last vector coincides with the initial point of the first vector, then the sum of vectors is a null vector (a vector with zero magnitude).

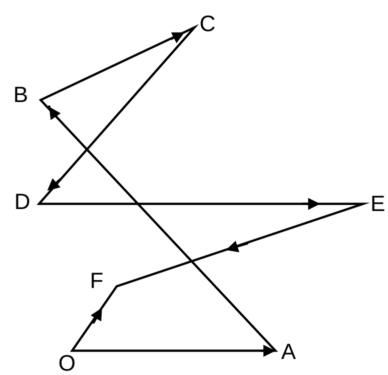


Illustration 4

Question: **If the vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a regular hexagon, express the other sides as vectors in terms of $\vec{a}$ and $\vec{b}$.**

Solution: ABCDEF is a regular hexagon.

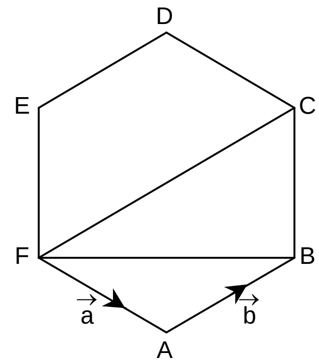
Let $\overrightarrow{FA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$.

$\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

$\overrightarrow{FC} = 2\vec{b}$ ($\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and lengthwise doubled)

$\therefore \overrightarrow{BC} = \overrightarrow{FC} - \overrightarrow{FB} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$

$\overrightarrow{CD} = -\vec{a}$; $\overrightarrow{DE} = -\vec{b}$; $\overrightarrow{EF} = \vec{a} - \vec{b}$.



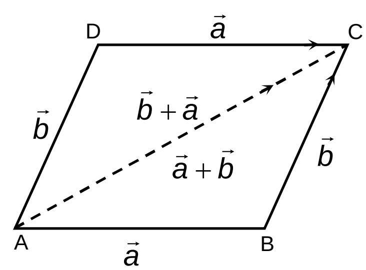
6. PROPERTIES OF VECTOR ADDITION

Property-1:

For any two vectors $\vec{a}$ and $\vec{b}$,

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Proof: Consider the parallelogram *ABCD*. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, then using the triangle law, from triangle *ABC*, we have $\overrightarrow{AC} = \vec{a} + \vec{b}$



Now, since the opposite side of a parallelogram are equal and parallel, from see the figure we have, $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$

Again using triangle law, from triangle *ADC*, we have

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$$

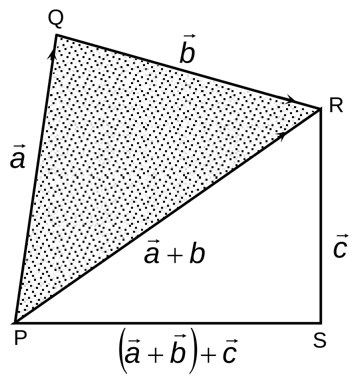
Hence $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Property-2:

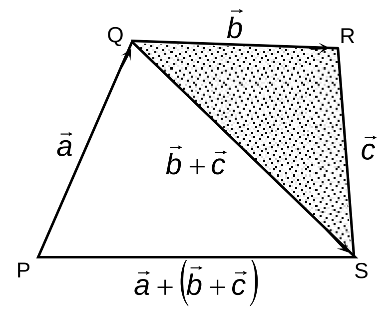
For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

Proof: Let the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be represented by $\overrightarrow{PQ}, \overrightarrow{QR}$ and $\overrightarrow{RS}$, respectively, as shown the figure.



(i)



(ii)

Then $\vec{a}+\vec{b}=\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$

and $\vec{b}+\vec{c}=\overrightarrow{QR}+\overrightarrow{RS}=\overrightarrow{QS}$

So, $(\vec{a}+\vec{b})+\vec{c}=\overrightarrow{PR}+\overrightarrow{RS}=\overrightarrow{PS}$

and $\vec{a}+(\vec{b}+\vec{c})=\overrightarrow{PQ}+\overrightarrow{QS}=\overrightarrow{PS}$

Hence $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

Remark:

The associative property of vector addition enables us to write the sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ as $\vec{a}+\vec{b}+\vec{c}$ without using brackets.

Note that for any vector $\vec{a}$, we have

$$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$$

Here, the zero vector $\vec{0}$ is called the additive identity for the vector addition.

7. MULTIPLICATION OF A VECTOR BY A SCALAR

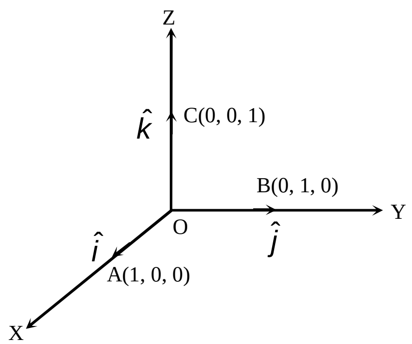
When a vector is multiplied by a scalar number, its magnitude gets multiplied but direction wise there is no change. Thus $k\overrightarrow{a}$ is a vector in the same direction of $\overrightarrow{a}$ but magnitude made k times. Thus if in the direction of $\overrightarrow{a}$, a unit vector is usually represented as $\hat{a}$ then $\overrightarrow{a} = |\overrightarrow{a}|\hat{a}$.

Thus any vector = (its magnitude) unit vector in that direction. It may be also said that $\overrightarrow{a}$ and $\hat{a}$ which are direction wise same, are collinear.

8. COMPONENTS OF A VECTOR

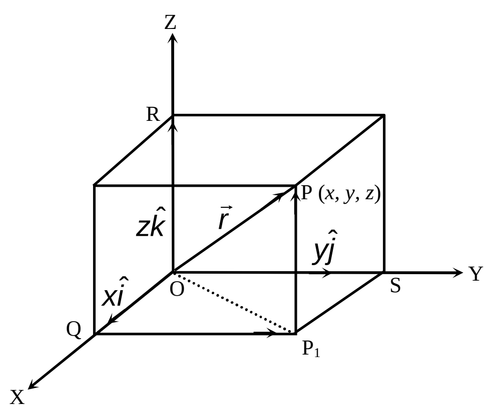
Let us take the points $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$ on the *x*-axis, *y*-axis and *z*-axis respectively. Then clearly

$$\left|\overrightarrow{OA}\right| - 1, \left|\overrightarrow{OB}\right| = 1 \text{ and } \left|\overrightarrow{OC}\right| = 1$$



The vector $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called unit vectors along the axes *OX, OY* and *OZ*, respectively, and denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.

Now, consider the position vector $\overrightarrow{OP}$ of a $P(x, y, z)$ as in figure. Let P_1 be the foot of the perpendicular from *P* on the plane *XOY*.



We, thus see that P_1P is parallel to *z*-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the *x, y* and *z*-axes, respectively and by the definition of the coordinates of *P*, we have

$$\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$$

Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\,\hat{i}$

Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$

and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}\text{) } = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its component form. Here x, y and z are called as the scalar components of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components or $\vec{r}$ along the respective axes. Sometimes x, y and z are also termed as rectangular components.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is readily determined by applying the Pythagoras theorem twice. We note that in the right angle triangle OQP_1.

$$\left|\overrightarrow{OP_1}\right| = \sqrt{\left|\overrightarrow{OQ}\right|^2 + \left|\overrightarrow{QP_1}\right|^2} = \sqrt{x^2 + y^2}$$

and in the right angle triangle OP_1P, we have

$$\overrightarrow{OP} = \sqrt{\left|\overrightarrow{OP_1}\right|^2 + \left|\overrightarrow{P_1P}\right|^2} = \sqrt{\left(x^2 + y^2\right) + z^2}$$

Hence the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$\left|\vec{r}\right| = \left|x\hat{i} + y\hat{j} + z\hat{k}\right| = \sqrt{x^2 + y^2 + z^2}$$

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(ii) the difference of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal if and only if
$a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$

(iv) the multiplication of vectors $\vec{a}$ by any scalar λ is given by
$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors and k and m be any scalars. Then

(i) $k\vec{a} + m\vec{a} = (k + m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

9. VECTOR JOINING TWO POINTS

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.

Joining the points P_1 and P_2 with the origin O, and applying triangle law, from the triangle OP_1P_2, we have $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$

Using the properties of vector addition, the above equation becomes

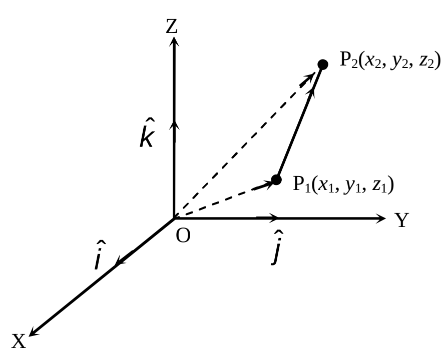
$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

i.e., $$\overrightarrow{P_1P_2} = \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

The magnitude of vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



10. SECTION FORMULA

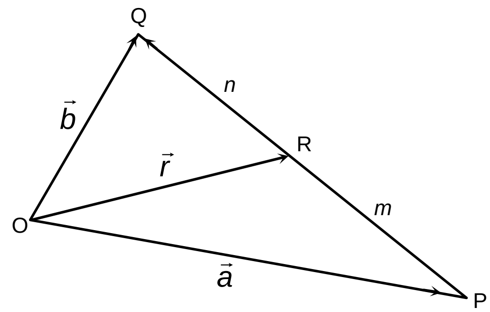
Let P and Q be two points represented by the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways – internally and externally. Here we intend to find the position vector $\overrightarrow{OR}$ for the point R with respect to the origin O. We take the two cases one by one.

Case-I: When R divides PQ internally.

If R divides $\overrightarrow{PQ}$ such that $m\overrightarrow{RQ} = n\overrightarrow{PR}$, when m and n are positive scalars, we say that the point R divides $\overrightarrow{PQ}$ internally in the ratio of $m:n$. Now from triangle ORQ and OPR, we have

$$\overrightarrow{RQ} = \overrightarrow{OQ} - \overrightarrow{OR} = \vec{b} - \vec{r}$$

and $$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{r} - \vec{a}$$

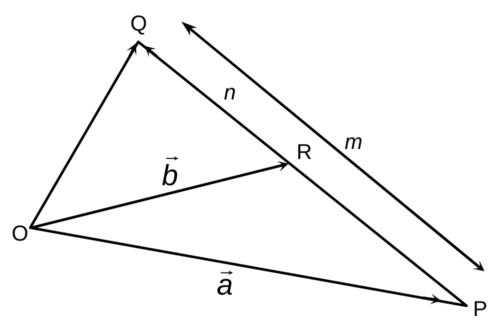


Therefore, we have $m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a})$ (why?)

or $$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Hence, the position vector of the point R which divides P and Q internally in the ratio of $m:n$ is given by $\overrightarrow{OR} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

Case-II: When r divides PQ externally. We leave it to the reader as an exercise to verify that the position vector of the point R which divides the line segment PQ externally in the ratio $m:n$ $\left(i.e.\ \dfrac{PR}{QR} = \dfrac{m}{n}\right)$ is given by



$$\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Remark:

If R is the mid-point of PQ, then $m = n$. And therefore, from Case-I, the mid-point R of $\overrightarrow{PQ}$, will have its position vector as

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$

11. PRODUCT OF TWO VECTORS

To meet the needs of physical situations we have two kinds of products of two vectors:

(i) Resulting in scalar value and used in situations of work done by a force in a displacement; projection of a given length on another direction. This product is usually termed dot product or scalar product.

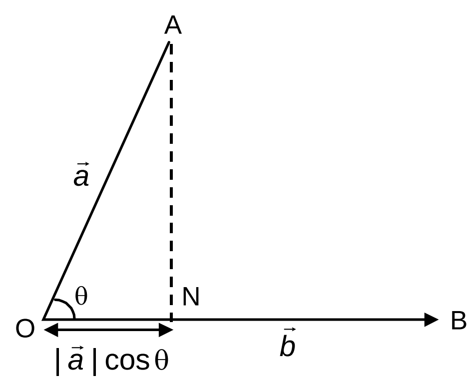
(ii) Resulting in a vector and used in situations of finding moment of a force about a point; or of finding the velocity of a particle having a rotating motion about a point with an angular velocity $\overrightarrow{\omega}$.

The two products with the related properties are stated:

11.1 THE SCALAR PRODUCT OR DOT PRODUCT

Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be any two vectors, forming between the two, an angle θ $(0 \le \theta \le \pi)$. Then the scalar product or dot product of $\overrightarrow{a}$ and $\overrightarrow{b}$ is denoted by $\overrightarrow{a} \cdot \overrightarrow{b}$ (Read $\overrightarrow{a}$ dot $\overrightarrow{b}$) and in value $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$

Geometrically, it represents projection of a vector on the other.



Properties of the Scalar product

1. $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}$ (Dot product is commutative)

2. $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \overrightarrow{c}$ (Dot product is distributive)

3. $\overrightarrow{a} \cdot \overrightarrow{b} = 0$ either when $|\overrightarrow{a}| = 0$ or when $|\overrightarrow{b}| = 0$ or when the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are orthogonal. Thus for any two perpendicular vectors the dot product vanishes.

4. $\cos\theta = \left|\dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}\right|$ where θ is the acute angle made by $\overrightarrow{a}$ with $\overrightarrow{b}$.

5. $\overrightarrow{a} \cdot \overrightarrow{a} = \overrightarrow{a}^2 = |\overrightarrow{a}|^2$

6. For the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{k} = \hat{k} . \hat{i} = \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{j} = 0$$

7. If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, *then* $\overrightarrow{a} \cdot \overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3$. so that

$$\cos\theta = \left| \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}} \right|$$

8. The projection of $\overrightarrow{a}$ on another direction represented by $\overrightarrow{b}$ is $\dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{b}|}$

9. If $\overrightarrow{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then

$$\overrightarrow{a} \cdot \hat{i} = a_1\,\hat{i} \cdot \hat{i} = a_1$$

Similarly $\overrightarrow{a} \cdot \hat{j} = a_2$ *and* $\overrightarrow{a} \cdot \hat{k} = a_3$ *so that*

$$\overrightarrow{a} = (\overrightarrow{a} \cdot \hat{i})\,\hat{i} + (\overrightarrow{a} \cdot \hat{j})\,\hat{j} + (\overrightarrow{a} \cdot \hat{k})\,\hat{k}$$

10. Work done by a force $\overrightarrow{F}$ in a displacement $\overrightarrow{AB}$ when the point of application of the force is displaced from *A* to *B*, is, $\overrightarrow{F} \cdot \overrightarrow{AB} = \overrightarrow{F} \cdot (\overrightarrow{OB} - \overrightarrow{OA}) = \overrightarrow{F}$. (position vector of *B* – position vector of *A*)

***Illustration* 6**

Question: **Use vector methods to prove, in the usual notation, in a triangle c^2 Z a^2 H b^2 J $2ab\cos C$.**

Solution: Let $\overrightarrow{AB} = \overrightarrow{c}, \overrightarrow{BC} = \overrightarrow{a}, \overrightarrow{CA} = \overrightarrow{b}$

So that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = 0$

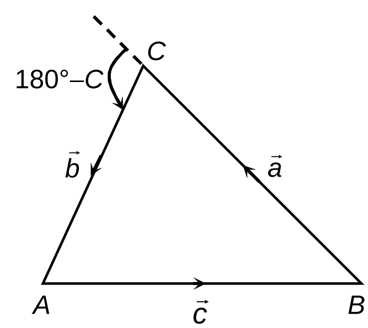
$$-\overrightarrow{c} = \overrightarrow{a} + \overrightarrow{b}$$

$\therefore \quad (-\overrightarrow{c}) \cdot (-\overrightarrow{c}) = (\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{a} + \overrightarrow{b})$

$\therefore \quad \overrightarrow{c}^2 = \overrightarrow{a} \cdot \overrightarrow{a} + \overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{a} + \overrightarrow{b} \cdot \overrightarrow{b}$

$\therefore \quad = \overrightarrow{a} \cdot \overrightarrow{a} + 2\overrightarrow{a} \cdot \overrightarrow{b} + \overrightarrow{b} \cdot \overrightarrow{b}$

$\overrightarrow{c}^2 = \overrightarrow{a}^2 + \overrightarrow{b}^2 + 2|\overrightarrow{a}||\overrightarrow{b}|\cos(180° - C)$



giving the result

12. SCALAR (OR DOT) PRODUCT OF TWO VECTORS

The scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined as $\vec{a}.\vec{b} = |\vec{a}| + |\vec{b}| \cos\theta$, where θ is the angle $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a}.\vec{b} = 0$

Observations:

1. $\vec{a}.\vec{b}$ is a real number.

2. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a}.\vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other i.e., $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

3. If $\theta = 0$, then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

   In particular $\vec{a}.\vec{a} = |\vec{a}|^2$, as θ in this case is 0.

4. If $\theta = \pi$, then $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

   In particular $\vec{a}.(-\vec{a}) = -|\vec{a}|^2$ as θ in this case is π.

5. In view of the observations 2 and 3, for mutually perpendicular unit vector $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ and $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

6. The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

   $$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \text{ or } \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

7. The scalar product is commutative i.e., $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (why?)

12.1 Two important properties of scalar product

Property-1:

(Distributivity of scalar product over addition over addition) Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$$

Property-2

Let $\vec{a}$ and $\vec{b}$ be any two vectors and λ be any scalar. Then

$$(\lambda\vec{a}).\vec{b} = (\lambda\vec{a}).\vec{b} = \lambda(\vec{a}.\vec{b}) = \vec{a}.(\lambda\vec{b})$$

12.2 Projection of a vector on a line

Suppose a vector $\overrightarrow{AB}$ makes an angle θ with a given directed line *l* (say), in the anticlockwise direction. Then the projection of $\overrightarrow{AB}$ on *l* is a vector $\vec{p}$ (say) with magnitude $\left|\overrightarrow{AB}\right|\cos\theta$, and the direction of $\vec{p}$ being the same (or opposite) to that of the line *l*, depending upon whether $\cos\theta$ is positive or negative. The vector $\vec{p}$ is called the projection vector and its magnitude $|\vec{p}|$ is simply called as the projection of the vector $\overrightarrow{AB}$ on the directed line *l*.

Observation:

1. If $\hat{p}$ is the unit vector along a line *l*, then the projection of a vector $\vec{a}$ on the line *l* is given by $\vec{a}.\hat{p}$.

2. Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}.\hat{b} \text{ or } \vec{a}.\left(\frac{\vec{b}}{|\vec{b}|}\right), \text{ or } \frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right)$$

3. If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

4. If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

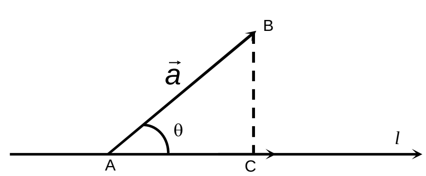
Remark:

If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos\alpha = \frac{\vec{a}.\hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|},\ \cos\beta = \frac{a_2}{|\vec{a}|} \text{ and } \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Also, note that $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively the projection of $\vec{a}$ along OX, OY and OZ i.e., the scalar components a_1, a_2, a_3 of the vector $\vec{a},$ are precisely the projections of $\vec{a}$ along *x*-axis, *y*-axis and *z*-axis, respectively. Further if $\vec{a}$ is a unit vector, then it may be expressed in terms of its direction cosines as

$$\vec{a} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$



(0° < θ < 90°)

13. THE VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be any two vectors forming an angle θ (0 ≤ θ < π). The vector product or cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ (Read as $\vec{a}$ cross $\vec{b}$); and this is a vector

(a) whose magnitude is $|\vec{a}||\vec{b}| \sin\theta$

(b) whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$ such that looked from this direction the rotation from $\vec{a}$ to $\vec{b}$ through an angle < π is anti-clockwise. It is written as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \cdot \hat{n}$ where $\hat{n}$ is a unit vector in the direction of $\vec{a} \times \vec{b}$ i.e. in the direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

Properties of the vector product

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (vector product is not commutative)

2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (vector product is distributive)

3. $\vec{a} \times \vec{b} = 0$ either when $|\vec{a}| = 0$ or when $|\vec{b}| = 0$ or when the vectors have the same direction. Thus the vector product between two collinear vectors is zero.

4. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $(\vec{a} \times \vec{b}) / |\vec{a} \times \vec{b}|$

5. $\vec{a} \times \vec{a} = 0$ for any vector $\vec{a}$

6. For the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ taken along the coordinate axes

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0} \text{ while}$$

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}; \quad \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \text{ and}$$

$$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

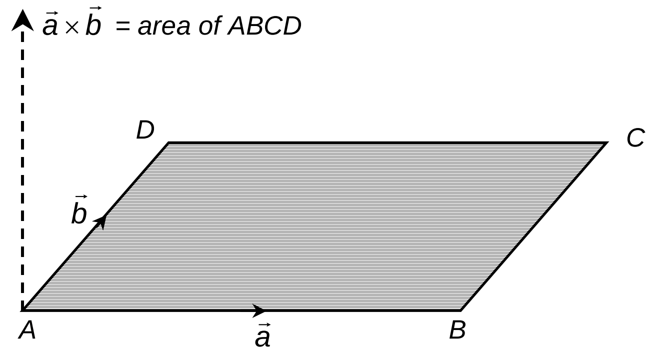
7. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $= \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$$

or in an equivalent determinant form,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8. $\vec{a} \times \vec{b}$ represents the vector area of the parallelogram whose adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



9. Let $\overrightarrow{F}$ be a force directed along a line. Let O be a point (origin). Let $\overrightarrow{OP} = \overrightarrow{r}$ be the position vector of any point P on the line of action of $\vec{F}$. Then $\vec{r} \times \vec{F}$ gives the moment of the force $\overrightarrow{F}$ about the point O.

10. Let $\overrightarrow{\omega}$ be the angular velocity of body rotating about an axis through O. If P be any point of the body with position vector $\overrightarrow{OP} = \overrightarrow{r}$, then $\overrightarrow{\omega} \times \overrightarrow{r}$ gives the velocity vector of P in the rotatory motion about the axis with an angular velocity $\overrightarrow{\omega}$.

***Illustration* 7**

Question: **For any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ prove that $(\overrightarrow{a} \cdot \overrightarrow{b})^2 + (\overrightarrow{a} \times \overrightarrow{b})^2 = |\overrightarrow{a}|^2 \, |\overrightarrow{b}|^2$**

Solution:

$$\left.\begin{aligned}(\overrightarrow{a} \cdot \overrightarrow{b})^2 &= |\overrightarrow{a}|^2 \, |\overrightarrow{b}|^2 \cos^2\theta \\ (\overrightarrow{a} \times \overrightarrow{b})^2 &= |\overrightarrow{a}|^2 \, |\overrightarrow{b}|^2 \sin^2\theta\end{aligned}\right\} \textit{where } \theta \text{ is the angle between } \overrightarrow{a} \text{ and } \overrightarrow{b}.$$

Adding, $(\overrightarrow{a} \cdot \overrightarrow{b})^2 + (\overrightarrow{a} \times \overrightarrow{b})^2 = |\overrightarrow{a}|^2 \, |\overrightarrow{b}|^2 \, (\cos^2\theta + \sin^2\theta)$

$$= |\overrightarrow{a}|^2 \, |\overrightarrow{b}|^2$$

PRACTICE PROBLEMS

PP1. If vector $\vec{a} + 2\vec{b} + \vec{c}$ and $2\vec{a} - \lambda\vec{b} - \mu\vec{c}$ are collinear find λ and μ.

PP2. If unit vector of $\hat{i} - \lambda\hat{j} + \hat{k}$ is $\frac{\hat{i} - \lambda\hat{j} + \hat{k}}{3}$ find λ.

PP3. *ABCD* is a quadrilateral. Forces $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$ act at a point. Show that their resultant is $2\overrightarrow{BA}$.

PP4. *ABCDE* is a pentagon, prove that

(i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = \vec{0}$

(ii) $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC} = 3\overrightarrow{AC}$

PP5. If the position vectors of *P* and *Q* be $(\hat{i} + 3\hat{j} - 7\hat{k})$ and $(5\hat{i} - 2\hat{j} + 4\hat{k})$ respectively, find $\overrightarrow{PQ}$ and $\overrightarrow{QP}$.

PP6. The adjacent sides of a parallelogram are represented by the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$. Find unit vectors parallel to the diagonals of the parallelogram.

PP7. If $A(a_1, a_2, a_3)$ is a given point and $\overrightarrow{AB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, find the coordinates of *B*.

PP8. Points *L, M, N* divide the sides *BC, CA, AB* of ΔABC in the ratio 1 : 4, 3 : 2, 3 : 7 respectively, prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, where *k* divides *AB* in the ratio 1 : 3.

PP9. If the points $A(m, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear, find the value of *m*.

PP10. Find the unit vector in the direction of $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find all the value of λ such that $(x, y, z) \neq (0, 0, 0)$ and

$$(\vec{i} + \vec{j} + 3\vec{k})x + (3\vec{i} + 3\vec{j} + \vec{k})y + (-4\vec{i} + 5\vec{j})z = \lambda(x\vec{i} + y\vec{j} + z\vec{k}).$$

Solution:

Collecting all the vectors to the L.H.S. in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$, we have

$$(x + 3y - 4z - \lambda x)\vec{i} + (x - 3y - 5z - \lambda y)\vec{j} + (3x + y - \lambda z)\vec{k} = 0$$

This is possible only if

$$(1-\lambda)x + 3y - 4z = 0$$

$$x + y(-3-\lambda) - 5z = 0$$

$$3x + y - \lambda z = 0$$

This has a non-trivial solution if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -3-\lambda & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

This simplifies to $\lambda(\lambda+1)^2 = 0$ which gives for λ the values 0, – 1, – 1

***Example* 2:**

Show that the vectors $\vec{a} = 3\vec{i} - 4\vec{j} + 2\vec{k}$; $\vec{b} = 4\vec{i} + 7\vec{j} + 8\vec{k}$

and $\vec{c} = 7\vec{i} + 3\vec{j} + 10\vec{k}$ form a right-angled triangle.

Solution:

$$\vec{a} + \vec{b} = 3\vec{i} - 4\vec{j} + 2\vec{k} + 4\vec{i} + 7\vec{j} + 8\vec{k}$$

$$= 7\vec{i} + 3\vec{j} + 10\vec{k} = \vec{c}$$

Hence $\vec{a}$, $\vec{b}$, $\vec{c}$ form the sides of a triangle.

Also $\vec{a} \cdot \vec{b} = (3\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (4\vec{i} + 7\vec{j} + 8\vec{k})$

$= 12 - 28 + 16 = 0$

Hence $\vec{a}$ and $\vec{b}$ are perpendicular. The triangle is therefore right-angled.

***Example* 3:**

If $\vec{a}$ and $\vec{b}$ are unit vectors, θ is the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$

Solution:

$\vec{a}\cdot\vec{a} = \vec{a}^{\,2} = |\vec{a}|^2 = 1;\quad \vec{b}\cdot\vec{b} = \vec{b}^{\,2} = |\vec{b}|^2 = 1$

Consider $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b}$

$= 1 + 1 - 2\cos\theta = 2(1 - \cos\theta)$

$= 4\sin^2\frac{\theta}{2}$

$\therefore |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$; giving $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} - \vec{b}|$

***Example* 4:**

Let $\vec{a} = 2\vec{i} + \vec{k}$; $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ *and* $\vec{c} = 4\vec{i} - 3\vec{j} + 7\vec{k}$. Determine a vector $\vec{x}$ such that, $\vec{x}\times\vec{b} = \vec{c}\times\vec{b}$ and $\vec{x}\cdot\vec{a} = 0$.

Solution:

It is given that $\vec{x}\times\vec{b} = \vec{c}\times\vec{b}$

Vectorially (pre) multiply both sides by $\vec{a}$.

$\vec{a}\times(\vec{x}\times\vec{b}) = \vec{a}\times(\vec{c}\times\vec{b})$

$(\vec{a}\cdot\vec{b})\vec{x} - (\vec{a}\cdot\vec{x})\vec{b} = (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}$ .... (i)

$\vec{a}\cdot\vec{b} = 3;\quad \vec{a}\cdot\vec{c} = 15;\quad \vec{x}\cdot\vec{a} = 0$

Substituting these values in (i) we get,

$3\vec{x} = 3(4\vec{i} - 3\vec{j} + 7\vec{k}) - 15(\vec{i} + \vec{j} + \vec{k})$

$= -3\vec{i} - 24\vec{j} + 6\vec{k}$, giving

$\vec{x} = -\vec{i} - 8\vec{j} + 2\vec{k}$

***Example* 5:**

If the vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a regular hexagon, express the other sides as vectors in terms of $\vec{a}$ and $\vec{b}$.

Solution: ABCDEF is a regular hexagon.

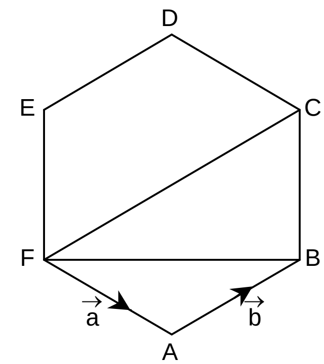
Let $\overrightarrow{FA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$.

$\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

$\overrightarrow{FC} = 2\vec{b}$ ($\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and lengthwise doubled)

$\therefore \overrightarrow{BC} = \overrightarrow{FC} - \overrightarrow{FB} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$

$\overrightarrow{CD} = -\vec{a}$; $\overrightarrow{DE} = -\vec{b}$; $\overrightarrow{EF} = \vec{a} - \vec{b}$.



***Example* 6:**

Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution: *ABC* is the triangle and *AD* is the median through *A*. If *AD* be produced to a length *DE* = *AD*, then *ACEB* is a parallelogram.

Hence by the parallelogram law of addition of two vectors,

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AE} = 2\overrightarrow{AD}$$

Similarly,

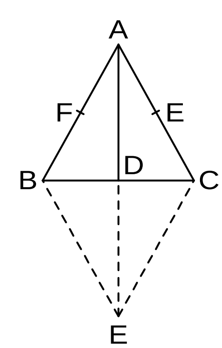
$\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BE}$ and $\overrightarrow{CB} + \overrightarrow{CA} = 2\overrightarrow{CF}$

Adding, we have

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CB} + \overrightarrow{CA} = 2\left(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}\right)$$

But the L.H.S is such that $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AB} - \overrightarrow{AB} = 0$.

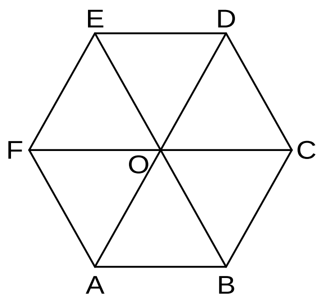
Similarly, the other two pairs also become zero. Hence $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \overrightarrow{0}$.



***Example* 7:**

Five forces represented by $\vec{AB}$, $\vec{AC}$, $\vec{AD}$, $\vec{AE}$ and $\vec{AF}$ act at the vertex A of a regular hexagon *ABCDEF*. Prove that their resultant is a force represented by 6 $\vec{AO}$, where O is the centre of the hexagon.

Solution:



$\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$

$= \vec{ED} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{CD}$

$= \vec{AC} + \vec{CD} + \vec{AE} + \vec{ED} + \vec{AD}$

$= \vec{AD} + \vec{AD} + \vec{AD} = 3\vec{AD}$

$= 6\,\vec{AO}$

This is the resultant required.

***Example* 8:**

Find the angle between $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 6\vec{i} - 3\vec{j} + 2\vec{k}$.

Solution:

Angle is θ given by $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$

$= = \dfrac{12 - 6 - 2}{\sqrt{9}\,\sqrt{49}} = \dfrac{4}{21}$

***Example* 9:**

Show that $\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$; $\vec{b} = 3\vec{i} - 6\vec{j} + 2\vec{k}$ *and* $\vec{c} = 6\vec{i} + 2\vec{j} - 3\vec{k}$ taken two by two are perpendicular.

Solution:

$\vec{a} \cdot \vec{b} = \left(2\vec{i} + 3\vec{j} + 6\vec{k}\right) \cdot \left(3\vec{i} - 6\vec{j} + 2\vec{k}\right)$

$= 6 - 18 + 12 = 0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular. Similarly $\vec{b}$ and $\vec{c}$ and $\vec{c}$ and $\vec{a}$ are perpendicular.

Note: Though $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$; no one of $|\vec{a}|, |\vec{b}|, |\vec{c}|$ is zero.

***Example* 10:**

If $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$ then prove that $|\vec{a}| = |\vec{c}|$.

Solution:

$\vec{a}.(\vec{a} \times \vec{b}) = \vec{a}.\vec{c} = 0$

Also $\vec{b}.(\vec{a} \times \vec{b}) = \vec{b}.\vec{c} = 0$, and $\vec{b}(\vec{b} \times \vec{c}) = \vec{a}.\vec{b} = 0$

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

$\Rightarrow$ $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| = |\vec{c}|$ and $|\vec{b}||\vec{c}| = |\vec{a}|$ $\Rightarrow$ $|\vec{b}| = 1$

EXERCISE – I

1. If $ABCD$ is a quadrilateral and E, F are the mid-point of AC and BD respectively, prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{EF}$.

2. The diagonals of a parallelogram $ABCD$ intersect each other at P. If O is any point, prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$.

3. Five forces $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AE}$ and $\overrightarrow{AF}$ act at the vertex of a regular hexagon $ABCDEF$. Prove that the resultant is $6\overrightarrow{AO}$, where O is the centroid of hexagon.

4. Show that the points A, B, C with position vectors $\vec{a}-2\vec{b}+3\vec{c}, 2\vec{a}+3\vec{b}-4\vec{c}$ and $-7\vec{b}+10\vec{c}$ are collinear.

5. Show that the points $2\hat{i}, -\hat{i}-4\hat{j}$ and $-\hat{i}+4\hat{j}$ form an isosceles triangle.

6. Prove that the points $\hat{i}-\hat{j}, 4\hat{i}+3\hat{j}+\hat{k}$ and $2\hat{i}-4\hat{j}+5\hat{k}$ are the vertices of a right-angled triangle.

7. Find the coordinates of the tip of the position vector which is equivalent to $\overrightarrow{AB}$, where the coordinates of A and B are (–1, 3,) and (–2, 1) respectively.

8. Show that the triangle ABC, the position vectors of whose vertices A, B, C are $7\hat{j}+10\hat{k}, -\hat{i}+6\hat{j}+\hat{k}$ and $-4\hat{i}+9\hat{j}+6\hat{k}$ respectively, is an isosceles right-angled triangle.

9. Find the vector from the origin O to the centroid of the triangle whose vertices are (1, –1, 2) (2, 1, 3) and (–1, 2, –1).

10. If $\vec{a}=(3\hat{i}-\hat{j}-4\hat{k}), \vec{b}=(-2\hat{i}+4\hat{j}-7\hat{k})$ and $\vec{c}=(\hat{i}+2\hat{j}-\hat{k})$, find a unit vector parallel to $(3\vec{a}-2\vec{b}+4\vec{c})$.

11. Show that the vectors $\vec{a}=3\sqrt{3}\,\hat{i}-3\hat{j}+2\hat{k}, \vec{b}=6\hat{j}-2\hat{k}$ and $\vec{c}=2\sqrt{3}\,\hat{i}+5\hat{j}\sqrt{3}\,\hat{k}$ form the sides of an equilateral triangle.

12. Prove that the vectors $(\hat{i}+2\hat{j}+3\hat{k}), (2\hat{i}+\hat{j}+3\hat{k})$ and $(\hat{i}+\hat{j}+\hat{k})$ are non-coplanar.

13. Prove that the points having position vectors $\hat{i}+2\hat{j}+3\hat{k}, 3\hat{i}+4\hat{j}+7\hat{k}, -3\hat{i}-2\hat{j}-5\hat{k}$ are collinear.

14. If $\vec{a},\vec{b}$ are two non-collinear vectors, prove that the points with position vectors $\vec{a}+\vec{b}, \vec{a}-\vec{b}$ and $\vec{a}+\lambda\vec{b}$ are collinear for all real values of λ.

15. Show that the vectors $\vec{a},\vec{b},\vec{c}$ given by $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ are non-coplanar. Express vector $\vec{d}=2\hat{i}-\hat{j}-3\hat{k}$ as a linear combination of the vectors $\vec{a},\vec{b}$ and $\vec{c}$.

EXERCISE – II

1. Let ABC is a triangle and D is the mid point of BC. Then find the value of $\overrightarrow{AB}+\overrightarrow{AC}-2\overrightarrow{AD}$.

2. $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are three vectors of magnitudes 3,4,5 units respectively. Also $\overrightarrow{a}$ is orthogonal to $\overrightarrow{b}+\overrightarrow{c}$, $\overrightarrow{b}$ *to* $\overrightarrow{c}+\overrightarrow{a}$ *and* $\overrightarrow{c}$ *to* $\overrightarrow{a}+\overrightarrow{b}$. Then find the length of the vector $\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}$.

3. If the scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector parallel to the sum of the vector $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1, then find the value of λ.

4. Find the value of $(\overrightarrow{i}+\overrightarrow{j})\cdot(\overrightarrow{j}+\overrightarrow{k})\times(\overrightarrow{k}+\overrightarrow{i})$.

5. Find the value of λ for which $2\overrightarrow{i}+\lambda\overrightarrow{j}+\overrightarrow{k}$ and $\overrightarrow{i}-2\overrightarrow{j}+4\overrightarrow{k}$ are perpendicular.

6. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then find the angle between $\vec{a}$ and $\vec{b}$.

7. Find a unit vector parallel to the sum of $2\overrightarrow{i}+4\overrightarrow{j}-5\overrightarrow{k}$ and $\overrightarrow{i}+2\overrightarrow{j}+3\overrightarrow{k}$.

8. Determine the unit vector perpendicular to the plane determined by $2\overrightarrow{i}-6\overrightarrow{j}-3\overrightarrow{k}$ and $4\overrightarrow{i}+3\overrightarrow{j}-\overrightarrow{k}$

9. Show that the sine of the angle between the two vectors $2\overrightarrow{i}-\overrightarrow{j}+\overrightarrow{k}$ and $3\overrightarrow{i}+4\overrightarrow{j}-\overrightarrow{k}$ is $\sqrt{\frac{155}{156}}$.

10. $ABCD$ is a parallelogram and E is the point of intersection of the diagonals. For any point O prove that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OE}$

11. Let $\overrightarrow{a}=a_1\overrightarrow{i}+a_2\overrightarrow{j}+a_3\overrightarrow{k}$; $\overrightarrow{b}=b_1\overrightarrow{i}+b_2\overrightarrow{j}+b_3\overrightarrow{k}$ and $\overrightarrow{c}=c_1\overrightarrow{i}+c_2\overrightarrow{j}+c_3\overrightarrow{k}$ be three non-zero vectors such that $\overrightarrow{c}$ is a unit vector perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$. If the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\pi/6$, then find the value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$.

12. If $\overrightarrow{A} = \overrightarrow{i} \times (\overrightarrow{a} \times \overrightarrow{i}) + \overrightarrow{j} \times (\overrightarrow{a} \times \overrightarrow{j}) + \overrightarrow{k} \times (\overrightarrow{a} \times \overrightarrow{k})$, then find vector $\vec{A}$.

13. Find the value of $\left|\overrightarrow{a} \times \overrightarrow{i}\right|^2 + \left|\overrightarrow{a} \times \overrightarrow{j}\right|^2 + \left|\overrightarrow{a} \times \overrightarrow{k}\right|^2$ for any vector $\overrightarrow{a}$.

14. $A_1, A_2, \ldots\ldots, A_n$ are the vertices of a regular plane polygon of n sides and O is its centre. Show that $\sum_{i=1}^{n-1} \overrightarrow{0A}_i \times \overrightarrow{OA}_{i+1} = (n-1)\left(\overrightarrow{OA}_n \times \overrightarrow{OA}_1\right)$

15. Let $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ be unit vectors such a that $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{a} \cdot \overrightarrow{c} = 0$ and the angle between $\overrightarrow{b}$ and $\overrightarrow{c}$ is $\frac{\pi}{6}$. Prove that $\overrightarrow{a} = \pm 2(\overrightarrow{b} \times \overrightarrow{c})$

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\lambda = -4$, $\mu = -2$

PP2. $\lambda = \pm\sqrt{7}$

PP5. $\overrightarrow{PQ} = 4\hat{i} - 5\hat{j} + 11\hat{k}$, $\overrightarrow{QP} = -4\hat{i} + 5\hat{j} - 11\hat{k}$

PP6. $\frac{1}{\sqrt{6}}(-\hat{i} + 2\hat{j} + \hat{k})$

PP7. $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

PP9. 1

PP10. $\hat{a} = \frac{\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)}{7}$

ANSWERS TO EXERCISE – I

7. (–1, –2)

9. $\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{4}{3}\hat{k}$

10. $\frac{17}{\sqrt{302}}\hat{i}-\frac{3}{\sqrt{302}}\hat{j}-\frac{2}{\sqrt{302}}\hat{k}$

ANSWERS TO EXERCISE – II

1. $\vec{0}$

2. 50

3. $\lambda = 1$

4. 2

5. $\lambda = 3$

6. 60°

7. $\frac{1}{7}(3\vec{i} + 6\vec{j} - 2\vec{k})$

8. $\frac{1}{7}(3\vec{i} - 2\vec{j} + 6\vec{k})$

11. $\frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

12. $\vec{A} = 2\vec{a}$

13. $2|\vec{a}|^2$