LESSON 9

DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Generally, any equation, such as

$$f(x, y, a) = 0 \quad \text{.... (i)}$$

represents for each individual value of *a*, a member of a family of curves. Sometimes it is found necessary to represent the whole family of curves as a single unit and consider them as one for the purpose of studying a common property or characteristic which may run through the members of the family.

From the given equation, solve for *a*, and the equation $\phi(x, y) = a$ may be obtained; and on differentiating, '*a*' gets removed. The resulting equation involving $\frac{dy}{dx}$ is known as a differential equation i.e. the equation representing all the members of the family $f(x, y, a) = 0$ or alternately $\phi(x, y) = a$.

2. DIFFERENTIAL EQUATION

An equation involving an independent variable *x*, a dependent variable *y* and the differential coefficients of the dependent variable i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ……etc is known as a differential equation. It can also be expressed as a function of variables *x*, *y* and derivatives of *y* w.r.t. *x* such as

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

Geometrically, differential equations represent a family of curves having a common property.

3. FORMATION OF DIFFERENTIAL EQUATION

To form a differential equation, we differentiate the given family of curves and eliminate the unknown constants as follows:

(i) Consider the equation $y = ax$. This represents the Cartesian equation to a family of straight lines through the origin.

Differentiating $y = ax$, we get $\frac{dy}{dx} = a$. Eliminating a, we get the differential equation $y = \frac{dy}{dx} \cdot x$.

Hence $y = x\frac{dy}{dx}$ is the differential equation of all straight lines passing through the origin.

(ii) Consider another example, the equation $x^2 + y^2 = a^2$. This, for various a, represents a family of concentric circles with centre at origin.

Differentiating the relation we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad (a \text{ is eliminated})$$

i.e. $$x + y\frac{dy}{dx} = 0$$

which may be said to be the differential equation to a family of concentric circles.

(iii) Now consider another equation representing a family of curves in the form

$$f(x, y, a, b) = 0 \qquad \text{.... (i)}$$

containing two arbitrary constants. In this case, since there are two constants, it becomes necessary to differentiate equation twice so that the result contains $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and can be expressed in the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \qquad \text{.... (ii)}$$

This equation is said to represent a differential equation of the family of curves represented by equation (i). Thus in the case of $y = ax + b$

$$\frac{dy}{dx} = a$$

$\frac{d^2y}{dx^2} = 0$ which is the differential equation of the family of all straight lines.

Illustration 1

Question: **Form the differential equation of the following relation:**

(i) $x^2 + y^2 = 2ax$ **(ii)** $x^2 + y^2 = 2ax + b$ **(iii)** $y = ae^x + be^{2x}$

Solution: (i) Consider the relation $x^2 + y^2 = 2ax$

Differentiating, $2x + 2y\frac{dy}{dx} = 2a$

Eliminating a, $x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$

$\Rightarrow \quad x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

In this case the relation contains only one constant and hence the differential equation contains only $\frac{dy}{dx}$.

(ii) Consider the relation $x^2 + y^2 = 2ax + b$

Differentiating $2x + 2y\frac{dy}{dx} = 2a$

Differentiating once again, $1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

which is the differential equation to the given equation and since there are two constants a and b, the differential equation contains (the second order) derivative $\frac{d^2y}{dx^2}$.

(iii) Consider the relation $y = ae^x + be^{2x}$

$$\frac{dy}{dx} = ae^x + 2be^{2x}$$

$$\frac{d^2y}{dx^2} = ae^x + 4be^{2x}$$

Consider $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = ae^x + 4be^{2x} - 3(ae^x + 2be^{2x}) + 2(ae^x + be^{2x})$

$\therefore$ For the relation $y = ae^x + be^{2x}$, we get the (second order) differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

4. ORDER AND DEGREE OF DIFFERENTIAL EQUATION

As we know that an equation containing an independent variable, a dependent variable and the derivatives of the dependent variable, is called a differential equation. It has an order and degree defined as follows:

4.1 ORDER OF A DIFFERENTIAL EQUATION

The highest derivative occurring in a differential equation defines its order.

4.2 DEGREE OF A DIFFERENTIAL EQUATION

The power of the highest order derivative occurring in a differential equation is called the degree of the differential equation, for this purpose the differential equation is made free from radicals and fractions of derivatives.

4.3 EXAMPLES

Differential equation	Order of D.E.	Degree of D.E.
• $\frac{dy}{dx}+4y=\sin x$	1	1
• $\left(\frac{d^2y}{dx^2}\right)^4+\left(\frac{dy}{dx}\right)^5-y=e^x$	2	4
• $\frac{d^2y}{dx^2}-\frac{dy}{dx}+3y=\cos x$	2	1
• $\frac{dy}{dx}=\frac{x^4-y^4}{xy\,(x^2+y^2)}$	1	1
• $y=x\frac{dy}{dx}+\sqrt{a^2\left(\frac{dy}{dx}\right)^2+b^2}$		
$\Leftrightarrow\ (x^2-a^2)\left(\frac{dy}{dx}\right)^2-2xy\,\frac{dy}{dx}+(y^2-b^2)=0$	1	2

- $\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}$

$\Leftrightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$ 2 2

5. GENERAL AND PARTICULAR SOLUTIONS OFA DIFFERENTIAL EQUATION

In earlier Classes, we have solved the equation of the type:

$x^2 + 1 = 0$ …(i)

and $\sin^2 x - \cos x = 0$ …(ii)

Solution of equation (i) and (ii) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown x in the given equation, LHS becomes equal to the RHS.

Now consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ ….(iii)

In contrast to the first two equations, the solution of this differential equation is a function ϕ that will satisfy it i.e., when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation LHS becomes equal to RHS.

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the given differential equation. Consider the function given by

$y = \phi(x) = a\sin(x + b)$ …(iv)

where $a, b \in R$. When this function and its derivative are substituted in equation (iii), LHS = RHS. So it is a solution of the differential equation (iii).

Let a and b be given some particular values say $a = 2$ and $b = \frac{\pi}{4}$, then we get a function

$y = \phi(x) = 2\sin\left(x + \frac{\pi}{4}\right)$

When this function and its derivative are substituted in equation (iii) again LHS=RHS.

Therefore ϕ_1 is also a solution of equation (iii).

Function ϕ consists of two arbitrary constants (parameters) *a, b* and it is called general solution of the given differential equation. Whereas function ϕ_1 contains no arbitrary constants but only the particular values of the parameters *a* and *b* and hence is called a particular solution of the given differential equation.

The solution which contains arbitrary constants is called the general solution (primitive) of the differential equation.

The solution free arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution of the differential equation.

***Illustration* 2**

Question: **Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.**

Solution:

Given function is $y = e^{-3x}$.

Differentiating both sides of equation with respect to *x*, we get

$$\frac{dy}{dx} = -3e^{-3x} \qquad \ldots(i)$$

Now, differentiating (i) with respect to *x*, we have $\frac{d^2y}{dx^2} = 9e^{-3x}$

Substituting the values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and *y* in the given differential equation, we get

LHS $= 9e^{-3x} + \left(-3e^{-3x}\right) - 6e^{-3x} = 9e^{-3x} - 9e^{-3x} = 0$ = RHS.

Therefore the given function is a solution of the given differential equation.

6. SOLUTION OF A DIFFERENTIAL EQUATION

6.1 EQUATIONS WITH SEPARABLE VARIABLE

Differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

can be reduced to form

$$\frac{dy}{dx} = g(x)\,h(y)$$

where it is possible to take all terms involving *x* and *dx* on one side and all terms involving *y* and *dy* to the other side, thus separating the variables and integrating.

***Illustration* 3**

Question: **Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$.**

Solution: Separating the variables

$$\frac{dy}{dx} = e^{-y}\,(e^x + x^2)$$

$e^y dy = (e^x + x^2)\,dx$, integrating, the solution is

$$e^y = e^x + \frac{x^3}{3} + A$$

$\Rightarrow \quad 3\,(e^y - e^x) = x^3 + C$ (C is an arbitrary constant)

Illustration 4

Question: **Find the order and degree of differential equation of all the parabolas whose axes are parallel to the *x*-axis and having a latus rectum *a*.**

Solution: Equation of required parabola's is

$$(y - \beta)^2 = a\,(x - \alpha)$$

Differentiating both sides w.r.*t*. *x*

$$\Rightarrow \quad 2\,(y - \beta)\,\frac{dy}{dx} = a$$

Again differentiating, w.r.*t*. *x*

$$2\,(y - \beta)\,\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \left(2\,(y - \beta)\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0 \Rightarrow \quad a\,\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$$

Thus order of differential equation is 2 and degree is 1.

6.2 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equations of the form

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$$

where $f_1\,(x, y)$ and $f_2\,(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equation.

It can also be written in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, by dividing both the functions by x^n where n is the degree of function.

To solve this equation, substitute

$$\frac{y}{x} = t \quad \text{or} \quad y = tx$$

$\Rightarrow \qquad \frac{dy}{dx} = t + x\frac{dt}{dx}$

Then the equation reduces to $t + x\frac{dt}{dx} = f(t)$ which can be easily reduced to variable separable as $\frac{dt}{f(t)-t} = \frac{dx}{x}$.

Illustration 5

Question: **Solve the differential equation** $\frac{dy}{dx} = \frac{x - y}{x + y}$

(Note: $x - y$, $x + y$ are homogeneous in x and y of degree one)

Solution: Taking $y = vx$,

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}$$

Substituting in the given equation

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1 - v - v - v^2}{1+v}$$

Now, separating the variables

$$\int \frac{1+v}{1-2v-v^2}\,dv = \int \frac{dx}{x} + A$$

$$-\frac{1}{2}\log(1 - 2v - v^2) = \log x + A$$

$\log[(1 - 2v - v^2)\,x^2] = \text{constant}$; $(1 - 2v - v^2)x^2 = C$.

$x^2 - 2xy - y^2 = C$ is therefore the solution where C is an arbitrary constant.

6.5 LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only or constants, is known as linear differential equation.

To solve this equation, we try to convert both sides as perfect differentials multiplying the equation by another function of x say $R(x)$.

Then $R(x)\frac{dy}{dx} + P(x)\,R(x)y = Q(x)\,R(x)$

This can be reduced to $\frac{d}{dx}(y\,R(x)) = Q(x)\,R(x)$

if $\qquad \frac{d}{dx}(R(x)) = P(x)\,R(x)$

$\Rightarrow \qquad P(x) = \frac{R'(x)}{R(x)}$

On integrating both sides.

$\Rightarrow \qquad \int P(x)\,dx = \log R(x) \quad \Rightarrow \qquad R(x) = e^{\int P(x)\,dx}$

This function is known as integrating factor, I.*F*. = $e^{\int Pdx}$.

The solution of differential equation is given by

$$y\,(I.F.) = \int Q(x)\,(I.F)\,dx$$

Illustration 6

Question: **Solve the differential equation** $x\frac{dy}{dx}$ Z y J $\cos\left(\frac{1}{x}\right)$.

Solution: Here, $x\frac{dy}{dx} - y = -\cos\left(\frac{1}{x}\right)$

$\therefore \qquad \frac{dy}{dx} - \frac{1}{x}.\,y = -\frac{1}{x}\cos\left(\frac{1}{x}\right)$; this is in the linear form.

Integrating factor $e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$.

Multiplying by the integrating factor,

$\frac{1}{x}.\frac{dy}{dx} - \frac{1}{x^2}y = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right) \quad$ or $\quad \frac{d}{dx}\left\{\frac{1}{x}.\,y\right\} = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

or $\qquad d\left(\frac{y}{x}\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$

or $\qquad \int d\left(\frac{y}{x}\right) = \int -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$

$\therefore \qquad \frac{y}{x} = \int \cos\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)$

or $\qquad \frac{y}{x} = \sin\left(\frac{1}{x}\right) + c$, where c is an arbitrary constant.

$\therefore \qquad y - x\sin\left(\frac{1}{x}\right) = cx$.

PRACTICE PROBLEMS

PP1. From the differential equation representing the family of curves $y = a\cos(x + B)$, where *A* and *B* are parameter.

PP2. Show that $y = x^2 + 2x + 1$ is the solution of the initial value problem

$$\frac{d^3y}{dx^3} = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 2 .$$

PP3. Solve: $(x+2)\frac{dy}{dx} = x^2 + 4x - 9,\ x \neq -2$

PP4. Solve: $\frac{dy}{dx} = \frac{x}{x^2+1}$

PP5. Solve: $\left(e^x + e^{-x}\right)\frac{dy}{dx} = \left(e^x - e^{-x}\right)$

PP6. Solve: $(x-1)\frac{dy}{dx} = 2xy$

PP7. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

PP8. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

PP9. Find the differential equation is the particular solution satisfying the given condition:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0;\ y = 2 \text{ when } x = 1$$

PP10. Find the differential equation is the particular solution satisfying the given condition:

$$\frac{dy}{dx} + 2y\tan x = \sin x;\ y = 0 \text{ when } x = \frac{\pi}{3}$$

PP11. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

PP12. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$, when *x* = 0.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find the equation of the curve passing through the point (–2, 3) given that the slope of the tangent to the curve at any point (*x, y*) is $\frac{2x}{y^2}$.

Solution:

We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

y $\frac{dy}{dx} = \frac{2x}{y^2} \Rightarrow y^2 dy = 2x\ dx$

On integrating both sides, we get $\int y^2 dy = \int 2x\ dx \Rightarrow \frac{y^3}{3} = x^2 + C$ …(i)

This equation represents the family of solution curves of given differential equation.
We have to find a particular member of this family which passes through the point $(-2, 3)$.

Substituting $x = -2$ and $y = 3$ in (i), we get $9 = 4 + C \Rightarrow C = 5$

Putting $C = 5$ in (i), we get $\frac{y^3}{3} = x^2 + 5$ as the equation of the required curve.

***Example* 2:**

In a bank principal increases at the rate of 5% per year. In how many years Rs. 1000 double itself.

Solution:

Let P be the principal at any time t. Then

$\frac{dP}{dt} = \frac{5P}{100} \Rightarrow \frac{dP}{dt} = \frac{P}{20} \Rightarrow \frac{1}{P} dP = \frac{1}{20} dt$

Integrating both sides, we get $\int \frac{1}{P} dP = \int \frac{1}{20} dt$

$\Rightarrow \log P = \frac{1}{20} t + \log C \Rightarrow \log \frac{P}{C} = \frac{1}{20} t \Rightarrow P = Ce^{t/20}$

When $t = 0$, we have $P = 1000$ …(i)

Substituting these values in (i), we get $P = 1000e^{t/20}$ ….(ii)

Let t_1 years be the time required to double the principal i.e., at $t = t_1,\ P = 2000$.

Substituting these values in (ii), we get $2000 = 1000e^{t_1/20}$

$\Rightarrow e^{t_1/20} = 2 \Rightarrow \frac{t_1}{20} = \log_e 2$

$\Rightarrow t_1 = 20 \log_e 2$

Hence, the principal doubles in $20 \log_e 2$

***Example* 3:**

Solve the differential equation is $\frac{dy}{dx} + \frac{y}{2x} = 3x^2, x > 0$.

Solution:

The given differential equation is $\frac{dy}{dx} + \frac{1}{2x} = 3x^2,\ x > 0$ …(i)

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q,$ where $P = \frac{1}{2x}$ and $Q = 3x^2$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int (1/2x)dx} = e^{(1/2)\log x} = e^{\log x^{1/2}} = x^{1/2}$

Multiplying both sides of (i) by I.F. = $\sqrt{x}$, we get

$\sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}}y = ex^{5/2}$

Integrating both sides w.r.t. x, we get

$y\sqrt{x} = \int 3x^{5/2}dx + C$

$\Rightarrow\ y\sqrt{x} = 3\left(\frac{x^{7/2}}{7/2}\right) + C \quad \Rightarrow\ y\sqrt{x} = \frac{6}{7}x^{7/2} + C$

Hence the required solution is given by $y\sqrt{x} = \frac{6}{7}x^{7/2} + C,\ x > 0$

***Example* 4:**

Solve $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ subject to the initial condition $y(0) = 0$.

Solution:

The given differential equation $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$ …(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where

$P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

We have $I.F. = e^{\int P\,dx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of (i) by $I.F.(1 + x^2)$, we get

$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Integrating both sides w.r.t. x, we get

$y(1+x^2) = \int 4x^2dx + C$

$\Rightarrow\ y(1+x^2) = \frac{4x^3}{3} + C$ …(ii)

It is given that y = 0, when x = 0. Putting x = 0 and y = 0 in (i), we get

$0 = 0 + C \ \Rightarrow\ C = 0$

Substituting $C = 0$ in (ii), we get

$y = \frac{4x^3}{3(1+x^2)}$, which is the required solution.

***Example* 5:**

Solve: $ydx - (x + 2y^2)dy = 0$.

Solution:

The given differential equation is $ydx - (x + 2y^2)dy = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x + 2y^2} \Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y} \Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y$ …(i)

This is a linear differential equation of the form

$\frac{dx}{dy} + Rx = S$, where $R = -\frac{1}{y}$ and $S = 2y$

$\therefore \ I.F. = e^{\int R dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$

Multiplying both sides of (i) by $I.F. y^{-1}$, we obtain

$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2$

Integrating both sides w.r.t. y, we get

$x.\frac{1}{y} = \int 2dy + C$

$\Rightarrow \frac{x}{y} = 2y + C$, which is the required solution.

***Example* 6:**

The surface area of a balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds, it is 5 units, find the radius after *t* seconds.

Solution:

Let r be the radius and S be the surface area of the balloon at any time t.

Then $S = 4\pi r^2$

$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ ….(i)

It is given that $\frac{dS}{dt}$ = constant = k (say). Putting $\frac{dS}{dt} = k$ in (i), we get

$k = 8\pi r \frac{dr}{dt}$

$\Rightarrow 8\pi r dr = kdt$

Integrating both sides, we get $4\pi r^2 = kt + C$

We are given that at $t = 0, r = 3$ and at $t = 2, r = 5$ …(ii)

$\therefore \ 36\pi = k(0) + C$ and $100\pi = 2k + C \Rightarrow C = 36\pi$ and $k = 32\pi$

Substituting the values of C and K in (ii), we obtain

$4\pi r^2 = 32\pi t + 36\pi \Rightarrow r^2 = 8t + 9 \Rightarrow r = \sqrt{8t + 9}$

***Example* 7:**

The slope of the tangent to the curve at any point is twice the ordinate at that point. The curve passes through the point (4, 3). Determine its equation.

Solution:

Let $P(x,\ y)$ be any point on the curve.

Then slope of the tangent at P is $\frac{dy}{dx}$.

It is given that the slope of the tangent at $P(x,\ y)$ is twice the ordinate i.e., $2y$

$\therefore\ \frac{dy}{dx} = 2y$

$\Rightarrow\ \frac{dy}{y} = 2dx\ \Rightarrow\ \log y = 2x + \log C\ \Rightarrow\ y = Ce^{2x}$ …(i)

Since the curve passes through (4, 3). Therefore $y = 3$ for $x = 4$

Putting $x = 4$ and $y = 3$ in (i), we get

$3 = Ce^{8}\ \Rightarrow\ C = 3e^{-8}$

Putting the value of C in (i), we get

$y = 3e^{2x-8}$

This is the required equation of the curve.

***Example* 8:**

Solve the differential equation $x\frac{dy}{dx}\ \text{H}\ \frac{y^2}{x}\ \text{Z}\ y$

Solution:

$\therefore\quad \frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$

On setting $y = vx$, the equation is $v + x\frac{dv}{dx} + v^2 = v$. Separating the variables and integrating,

$$\int \frac{dx}{x} + \int \frac{dv}{v^2} = A$$

$\log x - \frac{1}{v} = A$. This simplifies to the form $x = Ce^{x/y}$

***Example* 9:**

Solve the differential equation $(1\ \text{H}\ \cos 2x)\frac{dy}{dx}\ \text{J}\ (1\ \text{H}\ e^{y})\ \sin 2x = 0$ **; given that**

y **= 0, when** $x = \frac{\pi}{4}$

Solution:

$(1 + \cos 2x)\, dy = (1 + e^{y})\sin 2x\, dx$

Separating the variables

$$\frac{dy}{1+e^y} = \frac{\sin 2x}{1+\cos 2x}\,dx$$

$$\int\left[\frac{(1+e^y)-e^y}{1+e^y}\right]dy = \int \tan x\,dx$$

$$y - \log(1+e^y) + \log\cos x = A$$

$$y + \log\left(\frac{\cos x}{1+e^y}\right) = A$$

Taking $y = 0$, when $x = \frac{\pi}{4}$ we get $0 + \log\left(\frac{1}{2\sqrt{2}}\right) = A \Rightarrow A = -\frac{3}{2}\log 2$

The solution (called particular solution) is $y + \log\left[\left(\frac{\cos x}{1+e^y}\right)2\sqrt{2}\right] = 0$

$$\therefore 2\sqrt{2}\cos x = (1+e^y)\,e^{-y} = e^{-y} + 1$$

***Example* 10:**

Solve the differential equations

(i) $\mathbf{\frac{dy}{dx}\ \text{H}\ \frac{y^2\ \text{H}\ y\ \text{H}\ 1}{x^2\ \text{H}\ x\ \text{H}\ 1}\ \text{Z}\ 0}$ **(ii)** $\mathbf{\sqrt{4x\ \text{J}\ x^2}\ \frac{dy}{dx}\ \text{Z}\ 1\ \text{H}\ \cos 2y}$

Solution:

(i) $\frac{dy}{dx} = -\frac{(y^2+y+1)}{(x^2+x+1)}$

$$\frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

Integrating, $\int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + A$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + A$$

which may be also written as $\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = c$ …(i)

(where C is now arbitrary constant in which $\frac{2}{\sqrt{3}}$ is also absorbed)

The form (i) by combining the two terms on the LHS (using the formula

$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ may be reduced to

$2xy + x + y + C'(x + y + 1) = 1$

(ii) $\therefore \quad \frac{dy}{1+\cos 2y} = \frac{dx}{\sqrt{4x - x^2}}$

$\frac{1}{2}\int \frac{dy}{\cos^2 y} = \int \frac{dx}{\sqrt{4-(x-2)^2}} + C$

The solution is $\frac{\tan y}{2} = \frac{1}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$

or $\tan y = \sin^{-1}\left(\frac{x-2}{2}\right) + A$, where *'A'* is an arbitrary constant.

EXERCISE – I

1. Solve differential equation $\frac{dy}{dx}+2x=e^{3x}$.

2. Solve: $\frac{dy}{dx}=e^{x+y}$.

3. At any point (*x, y*) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

4. Solve the differential equation $x\frac{dy}{dx}+\cot y=0$, given that $y=\frac{\pi}{4}$ when $x=\sqrt{2}$.

5. Solve the differential equation is $\left(x^2-y^2\right)dx-2xy\,dy=0$.

6. Solve the differential equation is $x^2\frac{dy}{dx}=x^2+xy+y^2$.

7. Solve the differential equation is $\frac{dy}{dx}+\frac{y}{x}=\cos x+\frac{\sin x}{x},\ x>0$.

8. Solve: $\left(x+2y^3\right)dy=ydx$.

9. A population grows at the rate of 8% per year. How long does it take for the population to double? Use differential equation for it.

10. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.

11. The slope of the tangent to a curve at any point is reciprocal of twice the ordinate of that point. The curve passes through (4, 3). Formulate the differential equation and hence find the equation of the curve.

12. Solve differential equation $\left(x^2+1\right)\frac{dy}{dx}=1$.

13. Solve differential equation $\frac{dy}{dx}+\frac{1+y^2}{y}=0,\ y\neq 0$.

14. Solve differential equation $\frac{dy}{dx}=\sin^2 y$.

15. Solve: $\log\left(\frac{dy}{dx}\right)=ax+by$.

EXERCISE – II

1. Solve the differential equation $(1+e^{2x})dy+(1+y^2)e^x dx=0$ given that when $x=0,\ y=1$.

2. Solve: (i) $\sin^3 x \frac{dx}{dy}=\sin y$

(ii) $\frac{dy}{dx}+\frac{\sqrt{1-y^2}}{1-x^2}=0$

3. Solve the initial value problems $y-x\frac{dy}{dx}=2\left(1+x^2\frac{dy}{dx}\right),\ y(1)=1$

4. For the differentia equation $xy\frac{dy}{dx}=(x+2)(y+2)$. Find the solution curve passing through the point $(1,-1)$

5. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after *t* second.

6. Solve the differential equation is $(x^2-2xy)dy+(x^2-3xy+2y^2)dx=0$.

7. Solve the differential equation is $(2x^2y+y^3)dx+(xy^2-3x^3)dy=0$.

8. Solve the differential equation is $(\sin x)\frac{dy}{dx}+y\cos x=2\sin^2 x\cos x$.

9. Solve the differential equation is $(x^2-1)\frac{dy}{dx}+2(x+2)y=2(x+1)$.

10. The slope of the tangent at a point $P(x,y)$ on a curve is $\frac{-x}{y}$. If the curve passing through the point $(3,-4)$, find the equation of the curve.

11. Find the equation of the curve which passes through the point (2, 2) and satisfies the differential equation $y-x\frac{dy}{dx}=y^2+\frac{dy}{dx}$.

12. Prove that $x^2-y^2=c(x^2+y^2)^2$ is the general solution of differential equations $(x^3-3xy^2)dx=(y^3-3x^2y)dy$, where *c* is a parameter.

13. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$.

14. Find the particular solution of the differential equation $\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x\,dx=0$, given that $y=1$ when $x = 0$

15. Find a particular solution of the differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$, given that $y=0$ when $x = 0$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\frac{d^2y}{dx^2} + y = 0$, which is the required differential equation of the given family of curves.

PP3. $y = \frac{x^2}{2} + 2x - 13\log\left|x+2\right| + C,\ x \in R - \{2\}$

PP4. $y = \frac{1}{2}\log\left|x^2+1\right| + C,\ x \in R$

PP5. $y = \log\left|e^x + e^{-x}\right| + C,\ x \in R$

PP6. $2x + 2\log\left|x+1\right| = \log y + C$

PP7. $x^2 + y^2 = Cx$

PP8. $2y - 1 = e^x(\sin x - \cos x)$

PP9. $y = \frac{2x}{1 - \log|x|},\ (x \neq 0,\ x \neq e)$

PP10. $y = \cos x - 2\cos^2 x$

PP11. $y = 4 - x - 2e^x$

PP12. $4e^{3x} + 3e^{-4y} - 7 = 0$

ANSWERS TO EXERCISE – I

1. (i) Ans. $y + x^2 = \frac{1}{3}e^{3x} + C,\ x \in R$

2. (i) Ans. $-e^{-y} = e^x + C$

3. $y + 3 = (x + 4)^2$

4. $x = 2\cos y$

5. $x\left(x^2 - 3y^2\right) = C$

6. $\tan^{-1}\left(\frac{y}{x}\right) = C + \log|\, x\,|$

7. $y = \sin x + \frac{C}{x},\ x > 0$

8. $x = y^3 + cy$

9. $\frac{25}{2}\log 2$ years

10. $r = 3 - t$ and $0 \le t \le 3$

11. $y^2 = x + 5$

12. $y = \tan^{-1} x + c,\ x \in R$

13. $x + \frac{1}{2}\log\left|1 + y^2\right| + C$

14. $x + \cot y = C$

15. $\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C$

ANSWERS TO EXERCISE – II

1. $y = \frac{1}{e^x}$

2. (i) $\cos y - \frac{3}{4}\cos x + \frac{1}{12}\cos 3x = C,\ x \in R$

(ii) $y\sqrt{1-x^2} + x\sqrt{1-y^2} = C,\ x \in [-1, 1]$

3. $y = 2 - \frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$ is the required solution.

6. $\frac{y}{x} + \log x = C$

7. $x^2 y^{12} = C^4 \left| 2y^2 - x^2 \right|^5$

8. $y \sin x = \frac{2}{3}\sin^3 x + C$

9. $x^2 y = x^2 \sin x + 2x \cos x - \sin x + C$

10. $x^2 + y^2 = 25$

11. $2xy - 2x - y - 2 = 0$

13. $\sin^{-1} y + \sin^{-1} x = C$

14. $\tan^{-1} y + \tan^{-1}\left(e^x\right) = \frac{\pi}{2}$

15. $y = \log\left|\frac{2x+1}{x+1}\right|,\ x \neq 1$