LESSON 2

RELATIONS AND FUNCTIONS

1. RELATIONS

1.1 INTRODUCTION

In our day to day life, we often talk about relation between two persons, between two straight lines (e.g. perpendicular lines, parallel lines) etc.

Let A be the set of all male students in Delhi whose fathers live in Delhi. Let B be the set of all the people living in Delhi. Let a be a male student living in Delhi i.e. $a \in A$. Let b be the father of a. Then $b \in B$. And a is related to b under son-father relation. If we denote the son-father relation by symbol R then a is related to b under relation R. We can also express this by writing aRb. Here R denotes the relation 'is son of'.

We can also express this statement by saying that the pair of a and b is in relation R i.e., the ordered pair $(a, b) \in R$. This pair (a, b) is ordered in the sense that a and b can't be interchanged because first co-ordinate a represents son, and the second coordinate b represents father of a. Similarly if $a_1 \in A$ and b_1 is father of a_1, then $(a_1, b_1) \in R$. So we can think of the relation R as a set of ordered pairs whose first coordinate is in A and the second coordinate is in B. Thus $R \subseteq A \times B$. Since the relation 'is son of' i.e., R is a relation relating elements of A to be elements of B, we will say that R is a relation from set A to set B.

1.2 DEFINITION

A relation R, from a non-empty set A to another non-empty set B, is a subset of $A \times B$

Equivalently, any subset of $A \times B$ is relation from A to B.

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$

$$\Leftrightarrow R \subseteq \{(a, b) : a \in A,\ b \in B\}$$

Example: Let $A = \{1, 2\}, B = \{a, b, c\}$

Let $R = \{(1, a), (1, c)\}$

Here R is a subset of $A \times B$ and hence it is a relation from A to B.

2. DOMAIN AND RANGE OF A RELATION

2.1 DOMAIN OF A RELATION

Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in R$. Domain of R is precisely written as domain R.

Thus domain of (R) = $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$

Thus domain of R = set of first components of all the ordered pair which belong to R.

2.2 RANGE OF A RELATION

Let R be a relation from A to B. The range of R is the set of all those elements $b \in R$ such that $(a, b) \in R$ for some $a \in A$.

Thus range of $R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$.

Range of R = set of second components of all the ordered pairs which belong to R.

Set B is called as codomain of relation R.

Example1: Let $A = \{2, 3, 5\}$ and $B = \{4, 7, 10, 8\}$

Let $aRb \Leftrightarrow a$ divides b

Then $R = (2, 5)$ and range of $R = \{4, 10, 8\}$

Codomain of $R = B = \{4, 7, 10, 8\}$

Example2: Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$

Let R be a relation defined from A to B by $xRy \Leftrightarrow y$ is double of $x, \forall\, x \in A$

Then $1R2, 2R4, 3R6$

$\therefore\ R = \{(1, 2), (2, 4), (3, 6)\}$

3. REPRESENTATION OF A RELATION

A relation from a set *A* to set *B* can be represented in any one of the following four forms.

3.1 ROSTER FORM

In this form a relation *R* is represented by the set of all ordered pairs belonging to *R*.

Example: Let $A = \{-1, 1, 2\}$ and $B = \{1, 4, 9, 10\}$

Let *aRb* means $a^2 = b$

Then *R* (in roaster form) = $\{(-1, 1), (1, 1), (2, 4)\}$

3.2 SET-BUILDER FORM

In this form, the relation *R* is represented as $\{(a, b): a \in A, b \in B, a.....b\}$, the blank is to be replaced by the rule which associates *a* to *b*.

Example: Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

Let $R = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$, then *R* in the builder form can be written as

$R = \{(a, b): a \in A, b \in B; a - b = -1\}$

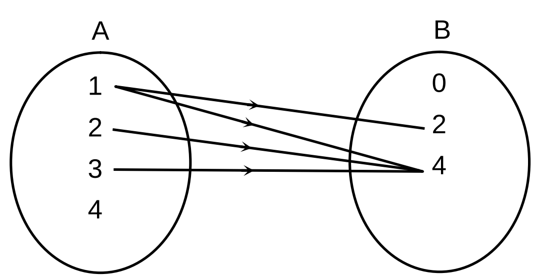
3.3 BY ARROW DIAGRAM

In this form, the relation *R* is represented by drawing arrows from first component to the second component of all ordered pairs belonging to *R*.

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{0, 2, 4\}$ and *R* be relation 'is less than' from *A* to *B*, then

$R = \{(1, 2), (1, 4), (2, 4), (3, 4)\}$

This relation *R* from *A* to *B* can be represented by the arrow diagram as shown in the figure.



4. TOTAL NUMBER OF RELATIONS

Let *A* and *B* be two non empty finite sets having *p* and *q* elements respectively.

Then $n(A \times B) = n(A).n(B) = pq$

Therefore, total number of subsets of $A \times B = 2^{pq}$

Since each subset of $A \times B$ is a relation from *A* and *B*, therefore total number of relations form *A* to *B* is 2^{pq}

Note: Empty relation ϕ and universal relation $A \times B$ are called trivial relations and any other relation is called a non trivial relation.

Example: Let $A = \{1, 2\}$, $B = \{3, 4, 5\}$

Then $n(A \times B) = n(A).n(B) = 2 \times 3 = 6$

$\therefore$ Number of relations from A to B = $2^6 = 64$

Illustration 1

Question: **If R is the relation 'is less than' from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the Cartesian product corresponding to R. Also find R^{-1} (aRb is a relation then bR' a is relation inverse to R i.e. $R' = R^{-1}$).**

Solution: Clearly, $R = \{(a, b) \in A \times B : a < b\}$

$\therefore$ $R = \{1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

So, $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$.

Illustration 2

Question: **Let $A = \{3, 5\}$, $B = \{7, 11\}$**

Let $R = \{(a, b) : a \in A, b \in B, a - b$ is even$\}$

Show that R is an universal relation from A to B.

Solution: Given, $= \{3, 5\}, B = \{7, 11\}$

Now, $= \{(a, b) : a \in A, b \in B$ and $a - b$ is even $\}$ = (3, 7), (3, 11), (5, 7), (5, 11)}

Also $A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Clearly, $R = A \times B$

Hence R is an universal relation from A to B.

Important formulae/points

- *If R is relation from A to B and (a, b) $\notin$ R, then we also write a $\not R$ b (read as a is not related to b)*
- *In an identity relation on A every element of A should be related to itself only.*
- *aRb shows that a is the element of domain set and b is the element of range set.*

PRACTICE PROBLEMS

PP1. Find the domain and range of the following relations:

(i) {(1, 2), (1, 4), (1, 6), (1, 8)} (ii) $\{(x, x^3) : x$ is a prime number less than 10}

PP2. Let A = (1, 2) and B = {3, 4}. Find the number of relations from A to B.

PP3. Let $R = \{(1, -1), (2, 0), (3, 1), (4, 2), (5, 3)\}$. Then

(i) write R in set builder form (ii) represent R by arrow diagram

5. FUNCTIONS

The concept of functions is very important because of its close relation with various phenomena of reality. Thus when we square a given real number in fact we perform an operation on the number x to get number x^2. Hence a function may be viewed as a rule which produces new elements from some given elements. Function is also called mapping or map.

- **Independent Variable**

The symbol which can take an arbitrary value from a given set is called an independent variable.

- **Dependent Variable**

The symbol whose value depends on independent variables is called a dependent variable.

6. DEFINITION OF A FUNCTION

- **Definition 1**

A function f is a relation from a non-empty set A to a non-empty set B such that domain of f is A and no two distinct ordered pairs in f have the same first element.

- **Definition 2**

Let A and B be two non-empty sets, then a rule of which associates each element of A with a unique element of B is called a mapping or a function from A to B we write $f : A \rightarrow B$ (read as f is a function from A to B).

If f associates $x \in A$ to $y \in B$, then we say that y is the image of the element x under the function f or the f image it by $f(x)$ and we write $y = f(x)$. The element x is called the pre-image or inverse-image of y.

Thus for a function from A to B:

(i) A and B should be non-empty.

(ii) Each element of A should have image in B.

(iii) No element of A should have more than one images in B.

Illustration 3

Question: **Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?**

(i) R = {(1, 2), (2, 2), (3, 1), (4, 2)}

(ii) R = {(2, 2), (1, 2), (1, 4), (4, 4)

(iii) R = {(1, 2), (2, 3), (4, 5), (5, 6), (6, 7)}

Solution: (i) Since first element of each ordered pair is different, therefore this relation is a function.

(ii) Since the same first element 1 corresponds to two different images 2 and 4, hence this relation is not a function.

(iii) Since first element of each ordered pair is different, therefore this relation is a function.

7. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

The set A is called as the domain of the map f and the set B is called as the co-domain. The set of the images of all the elements of A under the map f is called the range of f and is denoted by $f(A)$.

Thus range of f i.e. $f(A) = \{f(x) : x \in A\}$.

Clearly $f(A) \subseteq B$

Thus,

- It is necessary that every f image is in B, but there may be some elements in B, which are not f image of any element of A i.e., whose pre-image under f is not in A.
- Two or more elements of A may have same image in B.
- $f : x \to y$ means that under the function f from A to B, an element x of A has image y in B.
- If domain and range of a function are not to be written, sometimes we denote the function f by writing $y = f(x)$ and read it as y is a function of x.
- A function which has R or one of its subsets as its range is called "real valued function". Further, if its domain is also R or a subset of R, it is called a real function, where R is the set of real numbers.

8. IMPORTANT FUNCTIONS AND THEIR GRAPHS

- **Algebraic functions:** Functions consisting of finite number of terms involving powers and roots of the independent variable with the operations +, –, ×, ÷ are called algebraic functions.

  Examples: $f(x) = \sqrt{x-1}$, $f(x) = \sqrt{x} + x^3$

- **Polynomial functions:** $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots\ldots, a_n \in \mathbf{R}$ is said to be *a* polynomial function of degree *n*.

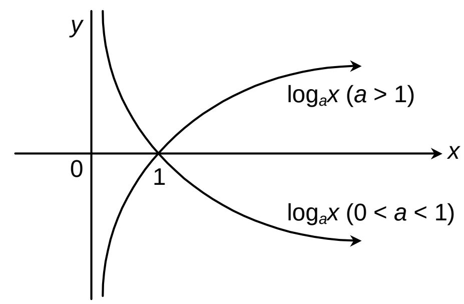
- **Logarithmic function:** If $a > 0$, $a \neq 1$, then the function $y = \log_a x$, $x \in R^+$ (set of positive real numbers) is called *a* logarithmic function, if $a = e$, the logarithmic function is denoted by $\ln x$.

  Logarithmic function is the inverse of the exponential function.

  For $\log_a x$ to be real, *x* must be greater than zero.

  $y = \log_a x$, $a > 0$ and $\neq 1$

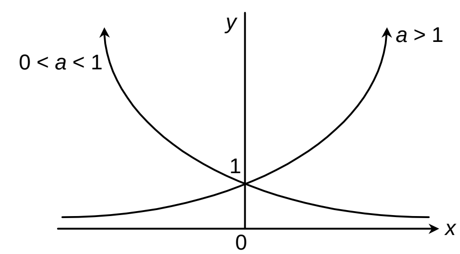
  Domain : $(0, \infty)$;   Range : $(-\infty, \infty)$;



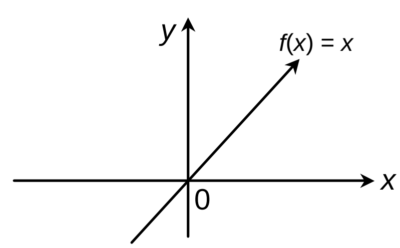
- **Exponential function:** If $a > 0$, $a \neq 1$, then the function defined by $y = a^x$, $x \in R$ is called an exponential function with base *a*.

  $y = f(x) = a^x$, $a > 0$, $a \neq 1$

  Domain : $\mathbf{R}$;   Range : $(0, \infty)$;



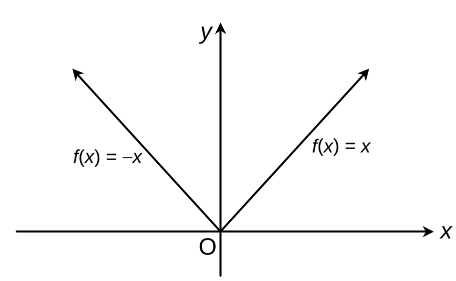
- **Identity function:** An identity function in *x* is defined as $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x$.



- **Absolute value function:** An absolute value function in *x* is defined as $f : \mathbf{R} \to \mathbf{R}$, $f(x) = |x|$.

  $$y = f(x) = |x| = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$$

  Domain : $\mathbf{R}$;   Range : $[0, \infty)$;



Note that $x = 0$ can be included either with positive values of *x* or with negative values of *x*. As we know, all real numbers can be plotted on the real number line, $|x|$ in fact represents the distance of number '*x*' from the origin, measured along the number-line. Thus $|x| \geq 0$. Secondly, any point 'x' lying on the real number line will have it's coordinates as (x, 0). Thus it's distance from the origin is $\sqrt{x^2}$. Hence $|x| = \sqrt{x^2}$. Thus we can define | x | as $|x| = \sqrt{x^2}$ e.g. if $x = -2.5$, then $|x| = 2.5$, if $x = 3.8$ then $|x| = 3.8$.

There is another way to define | x | as $|x| = \max\{x, -x\}$.

Basic properties of | *x* |

- $||x|| = |x|$
- Geometrical meaning of $|x - y|$ is the distance between x and y.
- $|x| > a \Rightarrow x > a$ or $x < -a$ if $a \in \boldsymbol{R}^+$ and $x \in \boldsymbol{R}$ if $a \in \boldsymbol{R}^-$.
- $|x| < a \Rightarrow -a < x < a$ if $a \in \boldsymbol{R}^+$ and $x \in \phi$ if $a \in \boldsymbol{R}^- \cup \{0\}$
- $|xy| = |x||y|$
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \neq 0$
- $|x + y| \leq |x| + |y|$

  It is *a* very useful and interesting property. Here the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). ($|x| + |y|$) represents the sum of distances of numbers x and y from the origin and $|x + y|$ represents the distance of number $x + y$ from the origin (or distance between 'x' and '$-y$' measured along the number line).

- $|x - y| \geq |x| \sim |y|$

  Here again the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). ($|x| - |y|$) represents the difference of distances of numbers x and y from the origin and $|x - y|$ represents the distance between 'x' and 'y' measured along the number line.

  The last two properties can be put in one compact form i.e., $|x| \sim |y| \leq |x \pm y| \leq |x| + |y|$.

➢ **Greatest integer function (step function):** The function $f(x) = [x]$ is called the greatest integer function and is defined as follows:

[x] is the greatest integer less than or equal to x .

Then $[x] = x$ if x is an integer

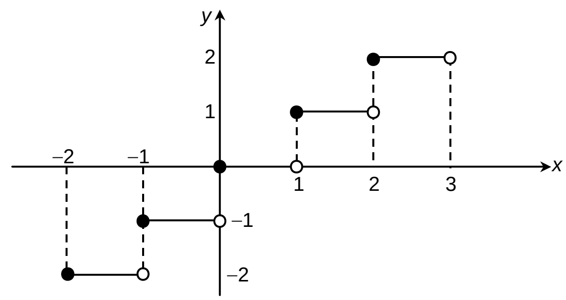
= integer just less than x if x is not an integer.

Examples: $[3] = 3$, $[2.7] = 2$, $[-7.8] = -8$, $[0.8] = 0$

In other words if we list all the integers less then or equal to x, then the integer greatest among them is called greatest integer of x. Greater integer of x is also called integral part of x.

$y = f(x) = [\,x\,]$

Domain : $\boldsymbol{R}$; Range : $\boldsymbol{I}$

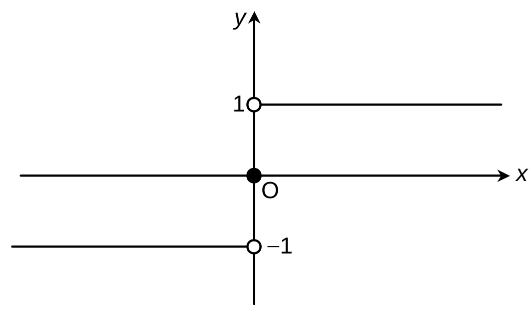


- **Signum function:** The function is defined as

  $y = f(x) = \text{sgn}\,(x)$

  $$\text{sgn}\,(x) = \begin{cases} \dfrac{|\,x\,|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

  or $$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

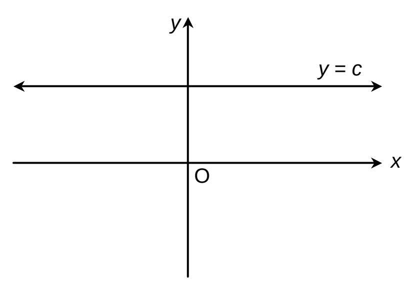


  Domain : $\boldsymbol{R}$; Range $\rightarrow$ {–1, 0, 1}

- **Rational algebraic function:** *A* function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called *a* rational function.

  The domain of *a* rational function $\dfrac{p(x)}{q(x)}$ is the set of all real numbers except points where $q(x) = 0$.

- **Constant function:** The function defined as f : $\boldsymbol{R} \rightarrow \{c\}$ where $f(x) = c$



9. ALGEBRAIC OPERATIONS ON FUNCTIONS

Let us consider two functions.

$f : D_1 \rightarrow \boldsymbol{R}$ and $g : D_2 \rightarrow \boldsymbol{R}$. We describe functions $f + g$, $f - g$, $f.g$ and f/g as follows:

- $f + g : D \to \boldsymbol{R}$ is *a* function defined by
  $(f + g)\, x = f(x) + g(x)$, where $D = D_1 \cap D_2$
- $f - g : D \to \boldsymbol{R}$ is *a* function defined by
  $(f - g)\, x = f(x) - g(x)$, where $D = D_1 \cap D_2$
- $f \,.\, g : D \to \boldsymbol{R}$ is *a* function defined by
  $(f \,.\, g)\, x = f(x) \,.\, g(x)$, where $D = D_1 \cap D_2$
- $f / g : D \to \boldsymbol{R}$ is *a* function defined by
  $(f / g)\, x = \dfrac{f(x)}{g(x)}$, where $D = D_1 \cap \{x \in D_2 : g(x) \neq 0\}$
- $(\alpha f)(x) = \alpha f(x),\ x \in D_1$ and α is any real number.

Illustration 4

Question: **If $f : R \to R$ is defined by $f(x) = x^3 + 1$ and $g : R \to R$ is defined by $g(x) = x + 1$, then find $f + g$, $f - g$, $f.g$, $\dfrac{f}{g}$ and $\alpha f\ (\alpha \in R)$.**

Solution: $f + g : R \to R$ is defined by $(f+g)(x) = f(x) + g(x) = x^3 + 1 + x + 1 = x^3 + x + 2$

$f - g : R \to R$ is defined by $(f-g)(x) = f(x) - g(x) = x^3 + 1 - x - 1 = x^3 - x$

$f.g : R \to R$ is defined by $(fg)(x) = f(x)g(x) = (x^3+1)(x+1) = x^4 + x^3 + x + 1$

$\dfrac{f}{g} : R - \{-1\} \to R$ is defined by $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^3+1}{x+1} = \dfrac{(x+1)(x^2-x+1)}{x+1} = x^2 - x + 1$

$\alpha f : R \to R$ is defined by

$(\alpha f)(x) = \alpha f(x) = \alpha(x^3+1) = \alpha x^3 + \alpha$

Illustration 5

Question: **Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$.**

Solution: Given $(f+g)(x) = \sqrt{x} + x$, $(f-g)(x) = \sqrt{x} - x$,

$(fg)\, x = \sqrt{x}(x) = x^{3/2}$ and $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{x}}{x} = x^{-1/2},\ x \neq 0$

10. TYPES OF FUNCTIONS

We have seen that f is a function from A to B, if each element of A has image in B and no element of A has more than one images in B.

But for a function *f* from *A* to *B* following possibilities are there

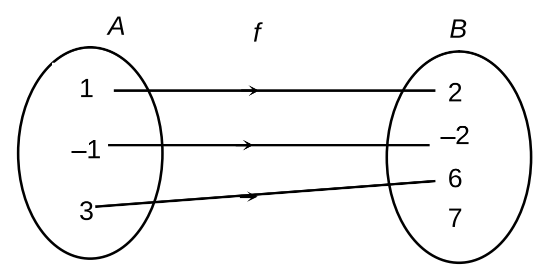
- Distinct elements of A have distinct images in B.
- More then one element of A may have same image in B.
- Each element of B is the image of some element of A.
- There may be some elements in B which are not the images of any element of A.

Because of the above mentioned possibilities, we have the following types of functions:

10.1 One-one or injective map

A map $f : A \to B$ is said to be one-one or injective if each and every element of set A has distinct images in set B.

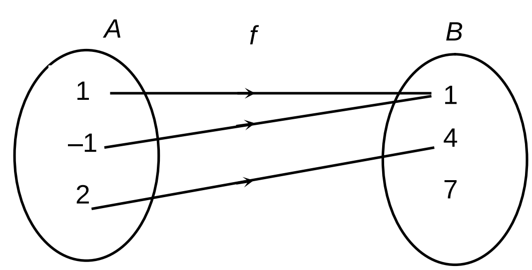
The map $f : A\{-1, 1, 3\} \to B\{-2, 2, 6, 7\}$ given by $f(x) = 2x$ is a one-one map.



10.2 Many one map:

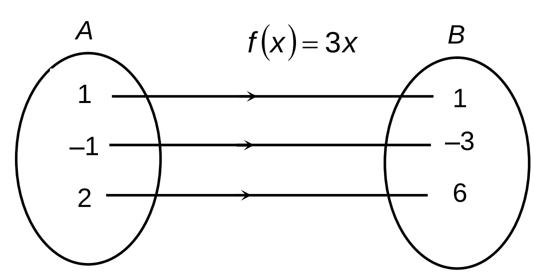
A map $f : A\{-1, 1, 2\} \to B\{1, 4, 7\}$ is said to be many one if and only if it is not one-one.

The map $f : A \to B$ given by $f(x) = x^2$ is a many-one map.



10.3 Onto map or surjective map:

A map $f : A \to B$ is said to be onto map or surjective map if and only if each element of B is the image of some element of A i.e. if and only if for every $y \in B$ there exists some $x \in A$ such that $y = f(x)$.



Thus *f* is onto iff $f(A) = B$ i.e. range of f = co-domain of *f*.

A map $f : A\{1, -1, 2\} \to B\{1, -3, 6\}$ given by $f(x) = 3x$ is an onto map.

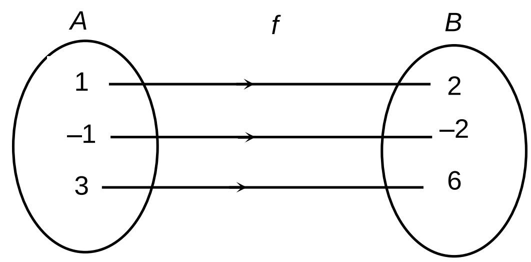
Note: Functions which are not onto, are into.

10.4 One-one onto map or bijective map:

A map $f : A \to B$ is said to be one-one onto or bijective if and only if it is both one-one and onto i.e., if

(i) distinct element of *A* have distinct images in *B*.

(ii) each element of *B* is the image of some element of *A*.



The map $f : A\{1, -1, 3\} \to B\{2, -2, 6\}$ given by $f(x) = 2x$ is a one-one onto map.

- A one-one onto function is also called a one-to-one correspondence or one-one correspondence.
- Let $f : A \to B$ be a function from finite set *A* to finite set *B*. Then

  1. *f* is one-one $\Rightarrow$ $n(A) \le n(B)$

  2. *f* is onto $\Rightarrow n(B) \le n(A)$

  3. *f* is one-one onto $\Rightarrow$ $n(A) = n(B)$

11. COMPOSITION OF FUNCTIONS

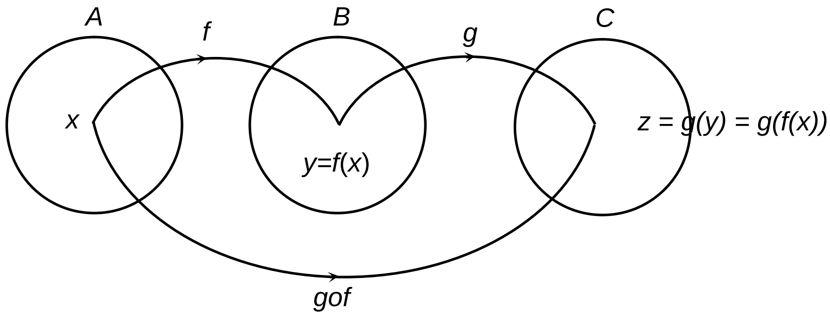
Let *A, B, C* be three non-empty sets, *f* be a function from *A* to *B* and *g* be a function from *B* to *C*. The question arises : can we combine these two functions to get a new function? Yes! The most natural way of doing this is to send every element $x \in A$ in two stages to an element of *C;* first by applying *f* to *x* and then by applying *g* to the resulting element $f(x)$ of *B* .

DEFINITION

Let $f : A \to B$ and $g : B \to C$ be any two mappings. Then *f* maps an element $x \in A$ to an element $f(x)$ = $y \in B$ and this *y* is mapped by *g* to an element $z \in C$. Thus $z = g(y) = g(f(x))$

Thus we have a rule, which associates with each $x \in A$, a unique element $z = g(f(x))$ of *C*. This rule is therefore a mapping from *A* to *C*. We denote this mapping by *gof* (read as '*g* composition f') and call it the composite mapping of *f* and *g*.

$$(gof)(x) = g(f(x)) \ \forall \ x \in A$$



The composition of two functions is also called the resultant of two functions or the function of a function.

Observe that the order of events occur from right to left i.e. *gof* reads composite of *f* and *g* and it means that we have to first apply *f* and then follow it up with *g*.

Note that for the composite function *gof* to exist, it is essential that range of *f* must be a subset of domain of *g*.

(i) Dom. (gof) = $\{x : x \in$ domain (f), $f(x) \in$ domain $(g)\}$

(ii) If *gof* is defined then it is not necessary that *fog* is defined.

Illustration 6

Question: **Let $f = \{(1, 2), (2, 3), (4, 5)\}$ and $g = \{(2, 3), (3, 5), (5, 2)\}$. Check whether *gof* and *fog* is defined, also find the range of *gof.***

Solution: $\because \ f = \{(1, 2), (2, 3), (4, 5)\}, \ g = \{(2, 3), (3, 5), (5, 2)\}$

Then dom . $f = \{1, 2, 4\}$; Range $f = \{2, 3, 5\}$; dom.$g = \{2, 3, 5\}$; Range $g = \{3, 5, 2\}$

since dom. g = Range f, $\therefore$ *gof* is defined

But dom . $f \neq$ Range g, $\therefore$ *fog* is not defined.

Also in this particular example, dom. (gof) = dom . $f = \{1, 2, 4\}$

$$(gof)(1) = g[f(1)] = g(2) = 3$$

$$(gof)(2) = g[f(2)] = g(3) = 5$$

$$(gof)(4) = g[f(4)] = g(5) = 2$$

Hence range of *gof* is {2, 3, 5}.

12. INVERSE FUNCTION

Let *f* be one-one and onto map from *A* to *B*. Since *f* is onto, therefore $\forall\ y \in B$ there exist $x \in A$ such that $f(x) = y$ and since *f* is one-one therefore this element *x* is unique. Thus we can define a map, say *g* from *B* onto *A* such that $g(y) = x$. This map *g* is called inverse map of *f* and is denoted by f^{-1}.

Thus $f^{-1} : B \to A$ such that $f^{-1}(y) = x$ iff $f(x) = y$

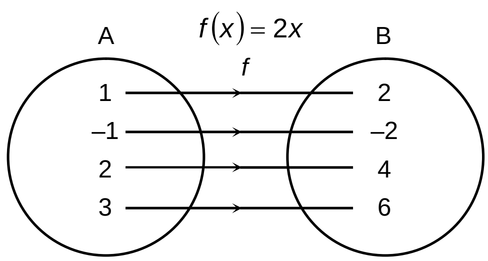
- **How to find the inverse of a given function?**

In order to find the inverse of the function $f(x)$, let $y = f(x)$

From this express *x* in terms of *y*. This value of *x* in terms of *y* will be $f^{-1}(y)$. Now put *x* in place of *y* in $f^{-1}(y)$ to get $f^{-1}(x)$.

Note: f^{-1} exists if and only if *f* is one-one onto.

Let $y = f(x) \Rightarrow y = 2x \Rightarrow x = \frac{y}{2} \Rightarrow f^{-1}(y) = \frac{y}{2} \Rightarrow f^{-1}(x) = \frac{x}{2}$



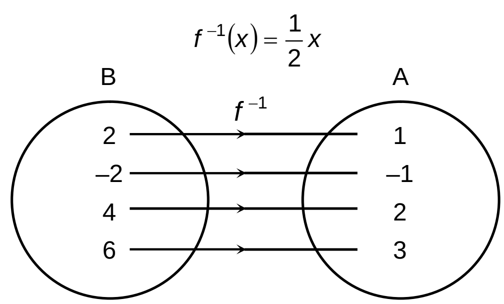


Illustration 7

Question: **Let *A* = {1, –1, 2, 3}, *B* = {2, –2, 4, 6}. The rule *f* given by $f(x) = 2x$ is a function from *A* and *B*. Give the mapping from *A* to *B*.**

Solution: Domain of $f = A = \{1, -1, 2, 3\}$, range of $f = \{2, -2, 4, 6\}$

Co-domain = {2, –2, 4, 6}

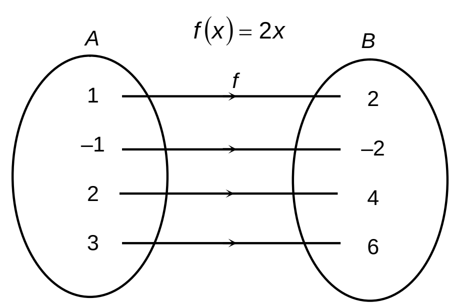


Illustration 8

Question: **Show that the map *f* : *A*¹J 1, 1, 2» ì *B*¹1, 4, 7» given by *f*Ɛ*x*Ƒ Z *x*² is not an onto map.**

Solution: ∵ Range of $f \neq$ co-domain of *f*

$\Rightarrow$ *f* is not an onto map.

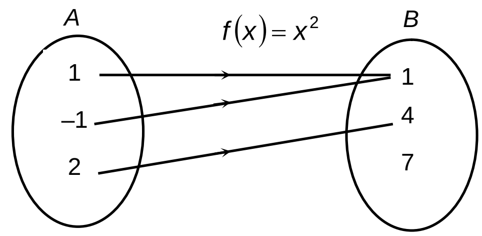


Illustration 9

Question: **Let *f* : *R* ì *R* be a function given by *f*Ɛ*x*Ƒ Z *ax* H *b* for all *x* Č *R*. Find the constants *a* and *b* such that *fof* Z *x*.**

Solution: Given, $f(x) = ax + b$ …(i)

Now, $fof = x$

$\Rightarrow \quad (fof)(x) = x$, for all $x \in R \quad \Rightarrow \quad f(f(x)) = x$, for all $x \in R$

$\Rightarrow \quad f(ax + b) = x$, for all $x \in R \quad \Rightarrow \quad a(ax + b) + b = x$, for all $x \in R$

$\Rightarrow \quad (a^2 - 1)x + ab + b = 0$, for all $x \in R$

Equating the coefficients of similar powers of *x*, we get,

$a^2 - 1 = 0$ and $ab + b = 0$

[∵ $(a^2 - 1)x + (ab + b) = 0$ is an identity in *x*]

$\Rightarrow \quad a = \pm 1$ and $b(a + 1) = 0$

When $a = 1, b(a + 1) = 0 \Rightarrow 2b = 0 \Rightarrow b = 0$

$\therefore \quad a = 1$ and $b = 0$ and when $a = -1$, $b(a + 1) = 0$, for all $b \in R$

$\therefore \quad a = -1$ and *b* may be any real number.

Hence, either *a* = 1 and *b* = 0 or $a = -1$ and $b \in R$

Illustration 10

Question: **Let *f* : *R* ì *R* and *g* : *R* ì *R* be defined by *f*Ɛ*x*Ƒ Z *x*², *g*Ɛ*x*Ƒ Z *x* H 2; ? *x* Č *R* (set of all real numbers), then find *gof* and *fog*. Is *gof* Z *fog* ?**

Solution: $(gof)(x) = g[f(x)] = g(x^2) = x^2 + 2$

$(fog)(x) = f[g(x)] = f(x + 2) = (x + 2)^2$

$(gof)(2) = 2^2 + 2 = 6$ and $(fog)(2) = (2 + 2)^2 = 16$. Hence $gof \neq fog$

Illustration 11

Question: **Let the function *f* : *R* ì *R* defined by *f*Ɛ*x*Ƒ Z 4*x* J 7 be one-one and onto. Find inverse of *f*(*x*).**

Solution: We have, $f(x) = 4x - 7,\ x \in R$

To find f^{-1}: $f(x) = y \Rightarrow 4x - 7 = y \Rightarrow x = \frac{y+7}{4}$

$\Rightarrow \quad f^{-1}(y) = \frac{y+7}{4} \qquad [\because f(x) = y \Leftrightarrow x = f^{-1}(y)]$

$\Rightarrow f^{-1}(x) = \frac{x+7}{4},\ x \in R$

Important formulae/points

Function f from A to B have the properties.

- *Distinct elements of A may have distinct images in B.*
- *More then one element of A may have same image in B.*
- *There may be some elements in B which are not the images of any element of A.*

PRACTICE PROBLEMS

PP4. Let $f : R \to R$ be defined by $f(x) = x^2$. Is *f* one-to-one?

PP5. Let *A* = {1, 2, 3}, *B* = {4, 5} and let *f* = {(1, 4), (2, 5), (3, 5)}, show that *f* is onto function from *A* to *B*.

PP6. If *A* = {1, 3, 5}, *B* = {2, 4, 6}, then find all possible one-one function from *A* to *B*.

PP7. Let *A* = {–1, 0, 1} and $f = \{(x, x^2) : x \in A)$. Show that $f : A \to A$ is neither one-to-one nor onto.

PP8. Let $f : R \to R$ be defined by $f(x) = 3x + 2$ is one-one and onto function. Find $f^{-1} : R \to R$.

PP9. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = x^2 + 2x - 3,\ g(x) = 3x - 4$, find $(gof)(x)$ and $(fog)(x)$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Solve $0 < |x-1| \le 3$ **for real values of *x*.**

Solution:

Here $|x-1| > 0$

$\Rightarrow \quad x \ne 1 \qquad \ldots(i)$

and $|x-1| \le 3 \Rightarrow -3 \le x-1 \le 3$

$\Rightarrow \quad -2 \le x \le 4 \qquad \ldots(ii)$

Combing (i) and (ii) $\Rightarrow \quad x \in [-2, 1) \cup (1, 4]$

***Example* 2:**

Solve $|x-1| + |2x-3| = |3x-4|$**.**

Solution:

Since $3x-4 = (x-1)+(2x-3)$

$\Rightarrow \quad |3x-4| = |x-1| + |2x-3|$

$\Rightarrow \quad (x-1)(2x-3) \ge 0$

$\Rightarrow \quad x \in (-\infty, 1] \cup \left[\frac{3}{2}, \infty\right)$

***Example* 3:**

Let $f(x) = \sqrt{x+3}$ **and** $g(x) = \sqrt{3-x^2}$ **, then find the domain for**

(i) $f+g$ **(ii)** $\frac{f}{g}$

Solution:

For domain of f : $x+3 \ge 0 \Rightarrow x \ge -3 \Rightarrow [-3, \infty) \qquad \ldots(i)$

For domain of g : $3-x^2 \ge 0$

$\Rightarrow \quad (3-x)(3+x) \ge 0$

$\Rightarrow \quad (x-3)(x+3) \le 0$

$\Rightarrow \quad x \in [-3, 3] \qquad \ldots(ii)$

(i) $f+g=\sqrt{x+3}+\sqrt{3-x^2}$

Domain of $(f+g)=$ domain of $(f)\cap$ domain of (g)

$=[-3,\ \infty)\cap[-3,\ 3]$

$=[-3,\ 3]$

(ii) For $\frac{f}{g}$

Here $g(x)\neq 0 \Rightarrow 3-x^2\neq 0 \Rightarrow x\neq \pm 3$

$\therefore \quad \frac{f}{g}=\frac{\sqrt{x+3}}{\sqrt{3-x^2}}=\frac{1}{\sqrt{3-x}}$

Domain of $\frac{f}{g}$ =domain of $(f)\cap$ domain of (g) $-\{-3,\ 3\}$

$=(-3,\ 3)$

***Example* 4:**

Find the domain of $f(x)=\frac{1}{\sqrt{x+|x|}}$ **.**

Solution:

$|x|$ is defined as

$$|x|=\begin{cases} x & , \quad x\geq 0 \\ -x & , \quad x<0 \end{cases}$$

$$\Rightarrow \quad x+|x|=\begin{cases} x+x=2x & , \quad x\geq 0 \\ x-x=0 & , \quad x<0 \end{cases}$$

$f(x)$ is defined for $x+|x|>0$

$\Rightarrow \quad x>0$

$\Rightarrow \quad x\in(0,\ \infty)$

$\therefore$ Domain of $f(x)$ is $(0,\ \infty)$

***Example* 5:**

Find the domain of the function $f(x)=\sqrt{\frac{1-|x|}{2-|x|}}$ **.**

Solution:

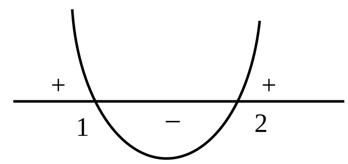
$f(x)=\sqrt{\dfrac{1-|x|}{2-|x|}}$

$f(x)$ is defined as $\dfrac{1-|x|}{2-|x|}\geq 0$ provided $|x|\neq 2 \;\Rightarrow\; x\neq \pm 2$ …(i)

$\Rightarrow \quad \dfrac{|x|-1}{|x|-2}\geq 0$

Let $|x|=t$

$\Rightarrow \quad \dfrac{t-1}{t-2}\geq 0$



$\Rightarrow \quad t\leq 1$ or $t\geq 2$

$\Rightarrow \quad |x|\leq 1$ or $|x|\geq 2$

$\Rightarrow \quad x\in[-1, 1]$ or $x\in(-\infty, -2]\cup[2, \infty)$ …(ii)

Combining (i) and (ii) $\Rightarrow$ domain of $f=(-\infty, -2)\cup(2, \infty)\cup[-1, 1]$

***Example* 6:**

Solve $\left|\dfrac{2}{x-3}\right|>1$**,** $x\neq 3$**.**

Solution:

$\left|\dfrac{2}{x-3}\right|>1 \quad\Rightarrow\quad \dfrac{2}{|x-3|}>1 \quad\Rightarrow\quad 2>|x-3| \quad\Rightarrow\quad |x-3|<2$

$\Rightarrow \quad -2<x-3<2 \quad\Rightarrow\quad 1<x<5$, but $x\neq 3 \quad\Rightarrow\quad x\in(1, 3)\cup(3, 5)$

***Example* 7:**

Solve $\dfrac{-1}{|x|-2}\geq 1,\;\; x\neq\pm 2$**.**

Solution:

$\dfrac{-1}{|x|-2}\geq 1 \quad\Rightarrow\quad \dfrac{-1}{|x|-2}-1\geq 0 \quad\Rightarrow\quad \dfrac{-1-(|x|-2)}{|x|-2}\geq 0$

$\Rightarrow \quad \dfrac{1-|x|}{|x|-2}\geq 0 \quad\Rightarrow\quad \dfrac{|x|-1}{|x|-2}\leq 0 \quad\Rightarrow\quad 1\leq |x|\leq 2 \quad\Rightarrow\quad x\in(-2, -1]\cup[1, 2]$ but $x\neq\pm 2$

$\Rightarrow \quad x\in(-2, -1]\cup[1, 2)$

***Example* 8:**

Find the domain of the function $y = f(x)$ **given by** $10^x + 10^y = 10$**.**

Solution:

$10^x + 10^y = 10 \quad \Rightarrow \quad 10^y = 10 - 10^x \quad \Rightarrow \quad y = \log_{10}(10 - 10^x) \Rightarrow \quad 10 - 10^x > 0$

$\Rightarrow \quad 10 > 10^x \Rightarrow x < 1$

$\Rightarrow$ Domain is $(-\infty, 1)$

***Example* 9:**

Find the domain of $f(x) = \log_5 \log_5 (1 + x^3)$**.**

Solution:

$f(x) = \log_5 \log_5 (1 + x^3)$

$\Rightarrow \quad \log_5(1 + x^3) > 0 \quad \Rightarrow \quad 1 + x^3 > 5^0 \quad \Rightarrow \quad 1 + x^3 > 1 \quad \Rightarrow \quad x^3 > 0$

$\Rightarrow \quad x \in (0, \infty)$ …(i)

Also $1 + x^3 > 0$

$x^3 > -1 \Rightarrow x > -1$ …(ii)

Combining (i) and (ii) $\Rightarrow$ Domain is $(0, \infty)$

***Example* 10:**

Find the domain of the function $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$**.**

Solution:

$f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2} = \dfrac{\log_2(x+3)}{(x+1)(x+2)}$

$\Rightarrow \quad x + 3 > 0 \Rightarrow x > -3$ …(i)

And $(x+1)(x+2) \neq 0 \Rightarrow x \neq -1, -2$ …(ii)

Combining (i) and (ii) $\Rightarrow x \in (-3, \infty) - \{-1, -2\}$

EXERCISE – I

1. Consider the graphs given below & state with reasons which of the following represents a function?

2. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find $A \times B$?

3. Let $f(x)=\begin{cases} 2-x & , \quad x<0 \\ 2 & , \quad x=0 \\ 2+x & , \quad x>0 \end{cases}$. Then consider the statements given below and state with reasons if they are correct or incorrect?

4. Find the range of $f(x)=x^2-7x+5$.

5. Find the domain of the function $f(x)=\sqrt{\frac{x-2}{x+2}}+\sqrt{\frac{1-x}{1+x}}$.

6. Find the domain for which the functions $f(x)=2x^2-1$ and $g(x)=1-3x$ are equal.

7. Let f be defined by $f(x)=x-3$ and g be defined by $g(x)=\begin{cases} \frac{x^2-9}{x+3} & , \quad x\neq -3 \\ k & , \quad x=3 \end{cases}$, then find the value of k such that $f(x)=g(x)$ for $\forall x \in R$.

8. Find the domain of the function $f(x)={}^6P_{x-3}$.

9. Let $f(x)=|x-1|$, then $f(|x|)=|f(x)|$ if $x \in A$.

Find the largest set A for which above statement is true.

10. If $f(x)$ is defined on [0, 1], then find the domain of $f(3x^2)$.

11. If $3f(x)-f\left(\frac{1}{x}\right)=\log x^4$, then find $f(e^{-x})$.

12. If $f_1(x)$ and $f_2(x)$ are defined on domain D_1 and D_2 respectively, then find $\text{dom}(f_1+f_2)\cap \text{dom}(f_1 f_2)$.

13. Find the domain of the function $f(x)=\frac{x^2+2x+1}{x^2-8x+12}$.

14. Find the range of the function $f(x)=|x-2|$.

15. Find the range of the function $f(x)=\frac{x^2}{1+x^2}$, $x \in R$.

EXERCISE – II

1. Find the domain and range of the following relations:

(a) $R_1 =$ {(1, 2), (1, 4), (1, 6), (1, 10)}

(b) $R_2 = \left\{\left(x, \frac{1}{x}\right): 0 < x < 4, x \text{ is an integer}\right\}$

2. Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B.

3. Let $A = \{a, b, c\}, B = \{x, y\}$. Find the total number of relations from A to B.

4. Which of the following relations are functions? Give reasons. If it is a function, find its domain and range.

(i) f = {(2, 1), (2, 3), (4, 3), (1, 2)}
(ii) g = {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}
(iii) h = {(–4, 4), (4, 4), (3, 2)}

5. Which of the following relations are function?

(i) $f = \{(x, y): y \text{ is the square root of } x : x \in R^+, y \in R\}$
(ii) $g = \{(x, y): e^y = x; x, y \in R\}$
(iii) $h = \{(x, y): y \text{ is the square root of } x; x, y \in R^+\}$
(iv) $k = \{(x, y): e^y = x; x \in R^+, y \in R\}$

6. The relation R_1 and R_2 are defined as

$$R_1(x) = \begin{cases} x^3 & ; \quad 0 \le x \le 4 \\ 4x^2 & ; \quad 4 \le x \le 6 \end{cases} \text{ and } R_2(x) = \begin{cases} x+2 & ; \quad -2 \le x \le 0 \\ 3x & ; \quad 0 \le x \le 6 \end{cases}.$$

Show that R_1 is a function and R_2 is not a function.

7. Let $f(x+1) = 3x+5$, find $f(x)$. Using definition of $f(x)$ complete the table given below:

x	–2	–1	0	1	2	3	4	7
$f(x)$								

Also draw the graph of $y = f(x)$.

8. A = {1, 2, 3, 5) and B = {4, 6, 9). Define a relation R from A to B by R = {(x, y): the difference between x and y is odd; $x \in A$, $y \in B$}. Write R in roster form.

9. Determine the domain and range of the relation R defined by
$R = \{(x,\ x+5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

10. Find the domain and range of following functions:
(i) $\sqrt{x-5}$ (ii) $|1-x|$

11. Let $f = \left\{\left(x, \frac{x^4}{1+x^4}\right) : x \in R\right\}$ be a function from R into R. Find the range of f.

12. Let $f, g : R \to R$ be defined, respectively by $f(x) = x+1,\ g(x) = 2x-3$. Find $f+g,\ f-g$ and $\frac{f}{g}$.

13. If $af(x) + bf\left(\frac{1}{x}\right) = x + \frac{5}{x},\ (a \ne b)$, then find $f(x)$.

14. Find the domain of the following functions:
(i) $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$
(ii) $f(x) = \log\left(\frac{\sqrt{16-x^2}}{3-x}\right)$

15. Find the domain of the following functions:

(i) $f(x) = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$
(ii) $f(x) = \sqrt{\log_{10}\frac{3-x}{x}}$

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

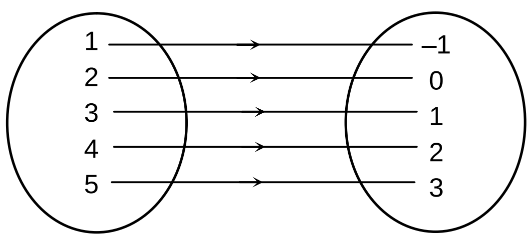
PP1. (i) Domain = {1}, Range = {2, 4, 6, 8}

(ii) Domain = {2, 3, 5, 7}, Range = {8, 27, 125, 343}

PP2 . 16

PP3. (i) $R = \{(a, b): a \in N, 1 \le a \le 5,\ b = a - 2\}$

(ii)



PP4. No

PP6. {(1, 2), (3, 4), (5, 6)}, {(1, 2), (3, 6), (5, 4)}, {(1, 4), (3, 2), (5, 6)}

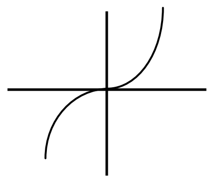
{(1, 4), (3, 6), (5, 2)}, {(1, 6), (3, 2), (5, 4)}, {(1, 6), (3, 4), (5, 2)}

PP8. $f^{-1}(x) = \frac{x - 2}{3}$

PP9. $(gof)(x) = 3(x^2 + 2x - 3) - 4 = 3x^2 + 6x - 13$

$(fog)(x) = (3x - 4)^2 + 2(3x - 4) - 3$

EXERCISE – I

1. 

2. {(1, 3), (1, 4), (2, 3), (2, 4)}

3. R, $[2, \infty)$

4. $\left[\frac{-29}{4}, \infty\right)$

5. ϕ

6. $\left\{-2, \frac{1}{2}\right\}$

7. –6

8. {3, 4, 5, 6, 7, 8, 9}

9. $[0, \infty)$

10. $\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

11. $-x$

12. $D_1 \cap D_2$

13. R

14. $[0, \infty)$

15. $[0, 1)$

EXERCISE – II

1. (a) Domain = {1}, Range = {2, 4, 6, 10} (b) Domain *R* = {1, 2, 3}, Range *R* = $\left\{1, \frac{1}{2}, \frac{1}{3}\right\}$

2. 16

3. 64

4. (i) Not a function as ordered pair (2, 1) and (2, 3) have the same first component.

(ii) It is a function, as first element of ordered pairs belongs to {2, 4, 6, 8, 10, 12, 14}, which are all distinct.

Domain of *g* = {2, 4, 6, 8, 10, 12, 14}

Range of *g* = {1, 2, 3, 4, 5, 6, 7}

(iii) It is a function.

Domain of *h* = {–4, 4, 3}

Range of *h* = {4, 2}

5. (i) *f* is a function (ii) *g* is not a function

(iii) *h* is a function (iv) *k* is a function

7. $f(x) = 3x + 2$

x	–2	–1	0	1	2	3	4	7
$f(x)$	–4	–1	2	5	8	11	14	23

8. *R* = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

9. Domain of *R* = {0, 1, 2, 3, 4, 5}

Range of *R* = {5, 6, 7, 8, 9, 10}

10. (i) Domain = $[5, \infty)$

Range = $[0, \infty)$

(ii) Domain = R

Range = $[0, \infty)$

11. Range = $[0, 1)$

12. $(f+g)x = 3x-2$; $(f-g)x = -x+4$ and $\left(\frac{f}{g}\right)x = \frac{x+1}{2x-3}, x \neq \frac{3}{2}$

13. $f(x) = \frac{1}{a^2-b^2}\left[x(a-5b)+\frac{1}{x}(5a-b)\right]$

14. (i) $[-1, 1]$

(ii) $(-4, 3)$

15. (i) $[-1, 2) \cup [3, \infty)$

(ii) $\left(0, \frac{3}{2}\right]$