LESSON 1

SETS

1. SET

A set is a collection of well defined distinct objects i.e. the objects follow a given rule or rules.

If we say that we have a collection of short students in a class, then this collection is not a set as short student is not well defined. If however, we say that we have a collection of students whose heights is less then 5 feet, then it represent a set.

Examples:

1. $A = \{1, 4, 5, 4, 8\}$, the elements of this collection are distinguishable but not distinct, hence A is not a set.

2. Let $A =$ collection of all vowels in English alphabets, then $A = \{a, e, i, o, u\}$. Hence elements of A are distinguishable as well as distinct, then A is a set.

3. The collection of all positive integers is a set.

4. The collection of all students of IIT (Delhi) is a set.

Some standard notation for some special sets:

1. The set of all natural number i.e., the set of all positive integers, is denoted by N.
2. The set of all integer number is denoted by I or Z.
3. The set of rational number is denoted by Q.
4. The set of all irrational number is denoted by Q'.
5. The set of all real number is denoted by R.
6. The set of all positive number is denoted by R^+. (zero is not included)
7. The set of all negative real number is denoted by R^-. (zero is not included)

8. The set of complex number is denoted by C.

1.1 REPRESENTATION OF SETS

- **Tabular form or Roster form**

In this method of describing a set, the elements of the set are listed separated by coma within braces.

Example: The set of prime number less then 10 can be described as {2, 3, 5, 7}

- **Set Builder form or Rule method**

In this method of describing a set, a variable x which stands for each element of the set is written under braces and then after giving a semicolor or oblique line the property or properties $P(x)$ possessed by each element of set is written the braces itself.

Example1: The set A of all even natural number can be written as $A = \{2x : x \in N\}$

Example2: The set $A = \{1, 3, 5\}$ can be written as $A = \{x : x$ is an odd natural number $\leq 5\}$

1.2 FINITE AND INFINITE SETS

A set having finite number of elements is called a **finite set**.

Example: $A = \{1, 2, 3, 4\}$. A is a finite set as it contains 4 elements.

A set which is not a finite set is called an **infinite set**. Thus a set A is said to be an infinite set if the number of elements of set A is not finite.

Example: Let A = set of all points on a particular straight line.

1.3 CARDINAL NUMBER OF A FINITE SET

The number of elements in a finite set A is called the cardinal number of set A and is denoted by $n(A)$

Example: Let $A = \{1, 2, 3, 4, 5\}$, then $n(A) = 5$

1.4 EQUIVALENT SETS

Two finite sets A and B are said to be equivalent if they have the same cardinal number. Thus set A and B are equivalent iff $n(A) = n(B)$.

If sets A and B are equivalent, we write $A \approx B$

Example: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, e, i, o, u\}$

Here $n(A) = n(B) = 5$

Therefore, sets A and B are equivalent.

1.5 EQUAL SETS

Two set A and B are said to be equal set if each element of set A is an element of set B and each element of B is an element of set A. Thus two sets A and B are equal if they have exactly the same elements. The order in which the elements in the two sets have been written is immaterial.

If set A and B are equal we can write A = B

Example1: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{x : x \in N$ and $1 \leq x \leq 5\}$

Here A and B are equal.

2. DIFFERENT TYPES OF SETS

2.1 NULL SET (OR EMPTY SET OR VOID A SET)

A set having no element is called null set or empty set or void set. It is denoted by ϕ or { }.

Example: The set of odd numbers divisible by 2.

2.2 SINGLETON SET

A set having single element is called a singleton set. It is represented by writing down the element within the braces.

Example: {2}, {0}, {ϕ}.

2.3 UNIVERSAL SET

A set consisting of all possible elements which occur in the discussion is called a universal set and is denoted by *U*.

2.4 PAIR SET

A set having two elements is called a pair set.

Example: {1, 2}, {2, 0}.

2.5 SET OF SETS

A set *S* having all its elements as set is called a set of sets or a family of sets or a class of sets.

Example1: $S = \{\{1, 2, 3\}, 3, \{4\}\}$ is not a set of sets as 3 is not a set.

Example2: {ϕ} is a singleton set of set having null set ϕ as its elements.

3. SUBSETS, SUPERSETS, PROPER SUBSETS

3.1 SUBSETS OF A SET

A set A is said to be a subset of a set *B* if each element of *A* is also an element of *B*. If *A* is a subset of set *B*, we write $A \subseteq B$

Thus, $A \subseteq B \;\Leftrightarrow\; [x \in A \;\Rightarrow\; x \in B]$

Example: Let *A* = {1, 2, 3}, *B* = {2, 3, 4, 1, 5}, then $A \subseteq B$.

The statement $A \subseteq B$ can also be expressed equivalently by writing $B \supseteq A$ (read '*B* is a superset of *A*')

If *A* is not a subset of *B* i.e., if there is an element in *A* which is not an element of *B,* then we write $A \not\subseteq B$ or $B \not\supseteq A$.

- **Some important properties of subset**
  - Every set is its own subset.

    Let A be any set ; $x \in A \Rightarrow x \in A$

    Hence $A \subseteq A$
  - Empty set is a subset of each set.
  - Let A and B be any two sets:

    then $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
  - Let A, B, C be three sets.

    If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

3.2 PROPER SUBSET OF A SET

A set A is said to be a proper subset of a set B, if A is a subset of B and $A \neq B$ i.e. if

Every element of A is an elements of B and B has at least one element which is not an element of A. This fact is expressed by writing $A \subset B$ or $B \supset A$.

If A is not a proper subset of B, then we write $A \not\subset B$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 1, 5\}$, then $A \subset B$ and $B \supset A$.

3.3 SUPERSET OF SETS

A set A is said to be a super set of set B, if B is a subset of A i.e., each elements of B is an elements of A. If A is a super set of B, then $A \supseteq B$.

Example: Let A = {1, 2, 3, 4, 5) and B = {2, 5, 4}.

Here B is a subset of A, therefore A is a superset of B.

3.4 POWER SET

The set or family of all the subsets of a given set A is said to be the power set of A and is denoted by $P(A)$

Example: If A = {1, 2}

$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

If A has n elements then $P(A)$ has 2^n elements.

Illustration 1

Question: **List all the subsets and all the proper subsets of the set {–1, 0, 1}.**

Solution: Let $A = \{-1, 0, 1\}$.

Subset of *A* having no element is : ϕ

Subsets of *A* having one element are : {–1}, {0}, {1}.

Subsets of *A* having two elements are : {–1, 0}, {0, 1}, {–1, 1}.

Subsets of *A* having three elements are : {–1, 0, 1}.

Thus, all the subsets of *A* are ϕ, {–1}, {0}, {1}, {–1, 0}, {0, 1}, {–1, 1}, {–1, 0, 1}.

Proper subsets of *A* are ϕ, {–1}, {0}, {1}, {–1, 0}, {0, 1}, {–1, 1}.

Illustration 2

Question: **Make correct statements by filling the blanks by suitable symbols $\subset$, $\not\subset$.**

(i) {*x* : *x* is an even natural number} _____ {*x* : *x* is an integer}
(ii) {*x* : *x* is a triangle in the plane} _____ {*x* : *x* is a rectangle in the plane}
(iii) {*x* : *x* is isosceles triangle in the plane} _____ {*x* : *x* is an equilateral triangle in the plane}
(iv) *a* _____ {*a*, {*b*}, *c*}
(v) {{*a*}} _____ {*a*, {*b*},*c*}

Solution:
(i) Since every even natural number is an integer, therefore,
{*x* : *x* is an even natural number} $\subseteq$ {*x* : *x* is an integer}.
(ii) Since a triangle is not a rectangle, therefore
{*x* : *x* is a triangle in the plane } $\not\subseteq$ {*x* : *x* is a rectangle in the plane}.
(iii) Since an isosceles triangle is not necessarily an equilateral triangle, therefore
{ *x* : *x* is an isosceles triangle} $\not\subseteq$ {*x* : *x* is an equilateral triangle}.
(iv) Since *a* is not a set, therefore, *a* $\not\subseteq$ {*a*, {*b*}, *c*}.
(v) Since {{*a*}} is a set containing exactly one element {*a*} and {*a*} is not an element of the set {*a*, {*b*}, *c*}, therefore, {{*a*}} $\not\subseteq$ {*a*, {*b*}, *c*}.

Illustration 3

Question: **How many elements are in the set**

$A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

$B = \{x : x$ is even integer and $x < 19\}$

$C = \{x : 0 \le x \le 1$ and x is a rational number$\}$

Solution: The elements of *A* are ϕ, $\{\phi\}$, $\{\phi, \{\phi\}\}$. So *A* has three elements.

$B = \{x : x = 0, \pm 2, \pm 4, \pm 6, \ldots$ and $x < 19\} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots, 18\}$

$\therefore$ *B* is an infinite set.

C is also infinite set because $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are all elements of *C*.

Important formulae/points

- *The order in which the elements of a set are written is immaterial thus the set {1, 2,3} and {2, 1, 3} are same.*
- *Two sets A and B are equal if* $x \in A \Rightarrow x \in B$ *and* $x \in B \Rightarrow x \in A$.
- *The set {0} is not an empty set as it contains one element 0.*
- *The set {¤} is not an empty set as it contain one element ¤.*
- $A \subseteq B \Rightarrow P(A) \subseteq P(B)$
- *If A has n elements then P(A) has* 2^n *elements.*

PRACTICE PROBLEMS

PP1. Which of the following collections are sets? Justify your answer.
(i) The collection of all girls in your class. (ii) The collection of all even integers.
(iii) The collection of beautiful girls of the world.

PP2. Write the following sets in set-builder form:
(i) $A = \{1, 4, 9, 16, \ldots\}$ (ii) $A = \{5, 9, 13, 17, 21, \ldots\}$
(iii) $A = \{14, 21, 28, 35, 42, \ldots, 98\}$ (iv) $A = \{3, 6, 9, 12, \ldots\}$

PP3. Write the following sets in tabular form (roster form) :
(i) $A = \{x : x$ is a natural number and $5 < x \leq 26\}$
(ii) $B = \{x : x^2 + x - 4 = 0\}$
(iii) $C = \{x : x = 2n - 1 < 20$ and $n \in N\}$
(iv) $D = \left\{x : x = \frac{n}{n+1}, n \in N\right\}$

PP4. Match each of the sets on the left described in the roaster form with the same set on the right described in the set-builder form:
(i) {H, A, R, Y, N } (a) $\{x : x$ is a natural number and a divisor of 18}
(ii) {L, I, T, E} (b) $\{x : x$ is a letter of the word HARYANA}
(iii) {1, 2, 3, 4, 6, 12} (c) $\{x : x$ is a letter of the word LITTLE }
(iv) {1, 2, 3, 6, 9, 18} (d) $\{x : x$ is a natural number and is a divisor of 12}

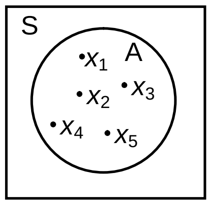
PP5. Determine the empty sets and, singleton set in the following sets:
(i) $A = \{x : x^2 = 2, x$ is a rational number} (ii) $B = \{x : x > 0$ and $x^2 = 25\}$
(iii) $C = \{x : x^3 + 1 = 0$ and x is an integer$\}$

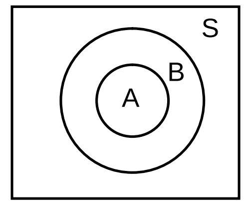
PP6. Find the number of subsets that can be formed from the set $A = \{4, 5, 6\}$.

PP7. If $A = \{1, 2, \{2, 3\}, 4\}$, state which of the following statements are true?
(i) $\{2, 3\} \subset A$ (ii) $\{2, 3\} \in A$
(iii) $\{2, 4\} \in A$ (iv) $\{\{2, 3\}\} \subset A$ (v) $\{1, 2, 3\} \subset A$

4. VENN DIAGRAMS

Statements involving sets can be easily understood with pictorial representation of the sets. A set is represented by circle or a closed geometrical figure A, inside the universal set *S,* which is represented by a rectangular region. Elements of a set *A* are represented by points within the circle which represents *A*.





$A \subseteq B$

4.1 OPERATION ON SETS

In algebra of numbers, the operation of addition (+) when applied on two numbers gives a third number $a+b$. Likewise we discuss the operation union ($\cup$), intersection ($\cap$) and difference (–) applicable on any two sets.

- **Union of two sets**

The union of two sets *A* and *B* is the set of all those elements which are either in *A* or in *B* or in both. This set is denoted by $A \cup B$ (read as '*A* union *B*').

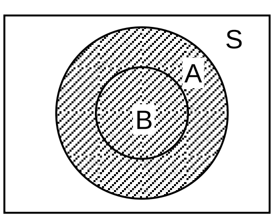
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$= \{x : x \in A \vee x \in B\} \qquad \{\vee \text{ denotes 'or'}\}$$

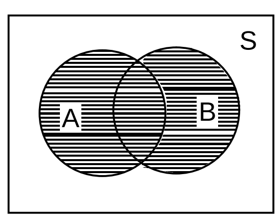
Also, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

Example: Let $A = \{1, 2, 3\}$ and $B = \{2, 1, 5, 6\}$, then $A \cup B = \{1, 2, 3, 5, 6\}$.

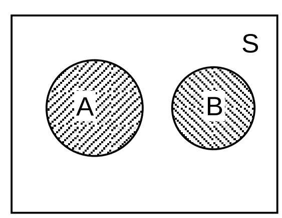
The union of two sets can be represented by Venn diagram as shown in the figure below:



$A \cup B$ when $A \subseteq B$



$A \cup B$ when neither $A \subseteq B$ nor $B \subseteq A$



$A \cup B$
when *A* and *B* disjoint sets

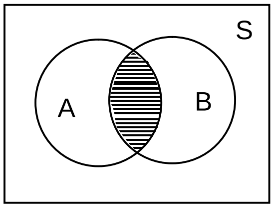
Here shaded portions are $A \cup B$.

- **Intersection of two sets**

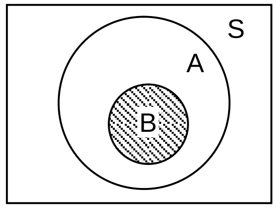
The intersection of two sets *A* and *B* is the set of all the elements which are common in *A* and *B*. This set is denoted as $A \cap B$ and read as *A* intersection *B*.

$$A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in A \wedge x \in B\} \quad \text{(^ denotes and)}$$

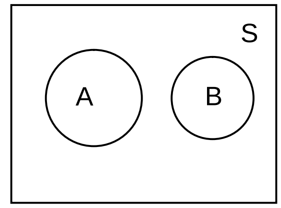
$$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$$



$A \cap B$ when neither $A \subseteq B$ nor $B \subseteq A$



$A \cap B$ when $B \subseteq A$, $A \cap B = B$



$A \cap B = \phi$ no shaded region when *A* and *B* are disjoint sets

Example: Let $A = \{1, 2, 3\}$ and $B = \{2, 1, 5, 6\}$, then $A \cap B = \{1, 2\}$

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

5. DIFFERENCE AND COMPLEMENTS

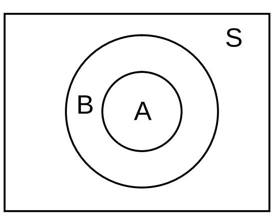
5.1 DIFFERENCE OF TWO SETS

The difference of two sets *A* and *B* is the set of all those elements of *A* which are not elements of *B*. It is denoted by $A - B$.

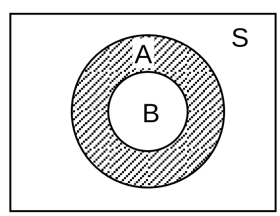
$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

thus $\quad x \in A - B \Leftrightarrow x \in A \text{ and } x \notin B$

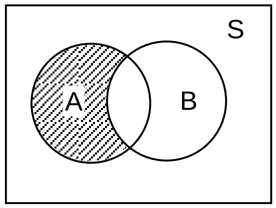
$A - B$ can be represented by Venn diagram (shaded region) as below:



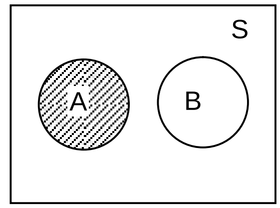
$A - B = \phi$ when $A \subseteq B$



$A - B$ when $B \subseteq A$



$A - B$ when neither $A \subseteq B$ nor $B \subseteq A$



$A - B = A$
when A and B are disjoint sets

Example: Let $A = \{1, 3, 5, 6, 7\}$; $B = \{2, 3, 4, 5\}$, then $A - B = \{1, 6, 7\}$; $B - A = \{2, 4\}$.

5.2 COMPLEMENT OF A SET

The complement of a set A is a set of all those elements of universal set S which are not elements of A. It is denoted by A^C or A'.

$$A' = S - A$$

6. LAWS OF ALGEBRA OF SETS

(i) Associative Law

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) Commutative Law

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$

(iii) Idempotent Law

- $A \cap A = A$
- $A \cup A = A$

(iv) Law of *U*

- $A \cap U = A$
- $A \cup U = U$

(v) Identity Law

- $A \cup \phi = A$
- $A \cap \phi = \phi$

(vi) Distributive Law

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(vii) De Morgan's Law

- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$

(viii) Complement Law

- $(A \cap A') = \phi$
- $(A \cup A') = U$
- $\phi' = U$
- $U' = \phi$

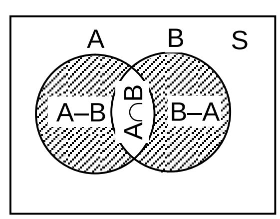
(ix) Involution Law (Law of double complementation)

- $(A')' = A$

7. SOME PRACTICAL APPLICATIONS OF SET THEORY

Here we shall study the use of set theory in practical problems.

The number of distinct elements of a finite set A is denoted by $n(A)$.



Use the following results whichever is required

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(iii) $n(A \cup B) = n(A) + n(B - A)$

(iv) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A \cap B = \phi$

(v) $n(A) = n(A - B) + n(A \cap B)$

(vi) $n(B) = n(B - A) + n(A \cap B)$

(vii) Number of elements belonging to exactly one of A and B

$= n(A - B) + n(B - A)$

$= n(A \cup B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

(viii) Number of elements belonging to exactly two of A, B and C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(ix) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(x) Number of elements belonging to exactly one of A, B and C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(xi) $n(A' \cap B') = n(S) - n(A \cup B)$ **=** $n(A \cup B)'$

(xii) $n(A' \cup B') = n(S) - n(A \cap B)$

Illustration 4

Question: **If *A* = {1, 3, 5, 6, 7}, *B* = {2, 3, 6, 8} and *C* = {1, 2, 3, 4}, then find**

(i) *A* ą *B* **(ii) *A* Ć *B*** **(iii) *A* – *B*** **(iv) *B* – *A***

Solution: (i) $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{3, 6\}$

(ii) $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{1, 2, 3, 5, 6, 7, 8\}$

(iii) $A - B = \{x : x \in A \text{ and } x \notin B\} = \{1, 5, 7\}$

(iv) $B - A = \{x : x \in B \text{ and } x \notin A\} = \{2, 8\}$

Illustration 5

Question: **If universal set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6\}$, $C = \{2, 3, 7\}$, then find (i) A' (ii) $(A - B)'$ (iii) $B' - A'$ (iv) $A' \cap B$ (v) $A \cup B'$ (vi) $(A \cap C)'$.**

Solution:

(i) $A' = \{x : x \in S \text{ and } x \notin A\} = \{0, 5, 6, 7, 8, 9\}$

(ii) $A - B = \{x : x \in A \text{ and } x \notin B\} = \{1, 4\}$

$\therefore \quad (A - B)' = S - (A - B) = S - \{1, 4\} = \{0, 2, 3, 5, 6, 7, 8, 9\}$

(iii) $B' = S - B = S - \{2, 3, 5, 6\} = \{0, 1, 4, 7, 8, 9\}$

$A' = S - A = S - \{1, 2, 3, 4\} = \{0, 5, 6, 7, 8, 9\}$

$B' - A' = \{0, 1, 4, 7, 8, 9\} - \{0, 5, 6, 7, 8, 9\} = \{1, 4\}$.

(iv) $A' \cap B = \{0, 5, 6, 7, 8, 9\} \cap \{2, 3, 5, 6\} = \{5, 6\}$

(v) $A \cup B' = \{1, 2, 3, 4\} \cup \{0, 1, 4, 7, 8, 9\} = \{0, 1, 2, 3, 4, 7, 8, 9\}$

(vi) $A \cap C = \{1, 2, 3, 4\} \cap \{2, 3, 7\} = \{2, 3\}$

$\therefore \quad (A \cap C)' = S - (A \cap C) = \{0, 1, 4, 5, 6, 7, 8, 9\}$

Illustration 6

Question: **If $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 8\}$, $C = \{2, 3, 4, 5, 6, 7\}$. Then verify that**

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (ii) $(A \cup B)' = A' \cap B'$

Solution:

(i) $B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A \cap B = \{1, 8\}$, $A \cap C = \{2, 4, 6\}$

Now, $A \cap (B \cup C) = \{x : x \in A \text{ and } x \in B \cup C\} = \{1, 2, 4, 6, 8\}$

$(A \cap B) \cup (A \cap C) = \{1, 2, 4, 6, 8\}$

$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(ii) $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$(A \cup B)' = \{x : x \in S \text{ and } x \notin A \cup B\} = \{9\}$

$A' = \{x : x \in S \text{ and } x \notin A\} = \{3, 5, 7, 9\}$

$B' = \{x : x \in S \text{ and } x \notin B\} = \{2, 4, 6, 9\}$

$A' \cap B' = \{9\}$

$\therefore \quad (A \cup B)' = A' \cap B'$.

Illustration 7

Question: **For any three sets *A*, *B*, *C*, prove the following (by using different laws on operations of sets):**

(i) ***A*** J ***B*** Z ***B*à** J ***A*à** **(ii)** **(*A*** Ć ***B*)** ą **(*A*** Ć ***B*à)** Z ***A***

(iii) ***A*** Ć ***B*** Z **(*A*** J ***B*)** Ć **(*B*** J ***A*)** Ć **(*A*** ą ***B*)**

Solution:

(i) $A - B = (A \cap B') = B' \cap A = B' \cap (A')' = B' - A'$.

(ii) $(A \cup B) \cap (A \cup B') = A \cup (B \cap B')$ [By distributive law]

$= A \cup \phi = A$

(iii) $(A - B) \cup (B - A) \cup (A \cap B) = [(A \cup B) - (A \cap B)] \cup (A \cap B)]$

$= [(A \cup B) \cap (A \cap B)'] \cup (A \cap B)$

$= [(A \cup B) \cup (A \cap B)] \cap [(A \cap B)' \cup (A \cap B)]$ [By distributive law]

$= (A \cup B) \cap S$, where S is the universal set

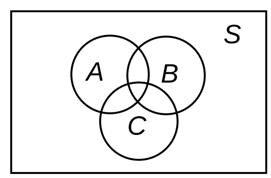
$= A \cup B$.

Illustration 8

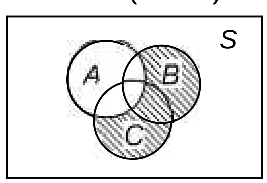
Question: **In the given figure shade the following sets:**

(i) ***A*à** ą **(*B*** Ć ***C*)**

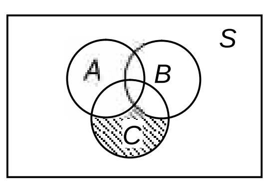
(ii) ***A*à** ą **(*C*** J ***B*)**



Solution: (i) Shaded region represents $A' \cap (B \cup C)$.



(ii) Shaded region represents $A' \cap (C - B)$



Important formulae/points

- $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$
- If $A \subseteq B$, then $A \cup B = B$
- $x \in A^C \Leftrightarrow x \in S$ and $x \notin A$
- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A' \cap B') = n(S) - n(A \cup B) = n(A \cup B)'$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- $n(B) = n(B - A) + n(A \cap B)$
- $n(A \cup B) = n(A) + n(B - A)$

PRACTICE PROBLEMS

PP8. For any three sets *A*, *B*, *C*, prove the following (by using different laws on operations of sets):

(i) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ (ii) $(A \cup B) - C = (A - C) \cup (B - C)$

PP9. Show that the following conditions are equivalent:

(i) $A \subseteq B$ (ii) $A \cap B = A$ (iii) $A - B = \phi$ (iv) $A \cup B = B$

PP10. If $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 8\}$, $C = \{2, 3, 4, 5, 6, 7\}$. Then verify that

(i) $A - (B \cup C) = (A - B) \cap (A - C)$ (ii) $A - C = A \cap C'$

PP11. If 63% of persons like orange where 76% like apples, what can be said about the % of persons who like both oranges and apples?

PP12. In a class of 25 students, 12 have taken mathematics, 8 have taken mathematics but not biology. Find the number of students who have taken both mathematics and biology and the number of those who have taken biology but not mathematics. Each student has taken either mathematics or biology or both.

8. CARTESIAN PRODUCT OF SETS

Let *a* be an arbitrary element of a given set *A* i.e. $a \in A$ and *b* be an arbitrary element of *B* i.e. $b \in B$. Then the pair (*a*, *b*) is an ordered pair. Obviously $(a, b) \neq (b, a)$. The Cartesian product of two sets *A* and *B* is defined as the set of ordered pairs (*a*, *b*). The Cartesian product is denoted $A \times B$.

$$\Rightarrow \quad A \times B = \{(a, b);\ a \in A,\ b \in B\}.$$

In general $A \times B \neq B \times A$ and if *A* or *B* is a null set, then $A \times B = \phi$.

Moreover, $n(A \times B) = n(A) \cdot n(B)$.

Note :

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) $(A - B) \times C = (A \times C) - (B \times C)$

(v) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(vi) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Illustration 9

Question: **If *A* = {2, 5}, *B* = {3, 4, 7} and *C* = {3, 4, 8} then evaluate *A* ò *B*, *B* ò *A*, *A* ò *A* and verify that**

(i) ***A* ò (*B* J *C*) Z (*A* ò *B*) J (*A* ò *C*)**

(ii) ***A* ò (*B* Ć *C*) Z (*A* ò *B*) Ć (*A* ò *C*)**

Solution: Here $A \times B = \{2, 5\} \times \{3, 4, 7\}$ = {(2, 3), (2, 4), (2, 7), (5, 3), (5, 4), (5, 7)}

$B \times A =$ {3, 4, 7} × {2, 5} = {(3, 2), (3, 5), (4, 2), (4, 5), (7, 2), (7, 5)} and

$A \times A$ = {2, 5} × {2, 5} = {(2, 2), (2, 5), (5, 2), (5, 5)}.

Also $A \times C$ = {2, 5} × {3, 4, 8} = {(2, 3), (2, 4), (2, 8), (5, 3), (5, 4), (5, 8)}

$B - C$ = {3, 4, 7} – {3, 4, 8} = {7}

$\Rightarrow$ $A \times (B - C)$ = {2, 5} × {7} = {(2, 7) (5, 7)}

$(A \times B) - (A \times C) =$ {(2, 3), (2, 4), (2, 7), (5, 3), (5, 4), (5,7) }

– {(2, 3), (2, 4), (2, 8), (5, 3), (5, 4), (5, 8)} = {(2, 7), 5, 7} = $A \times (B - C)$.

To verify (ii), we write $B \cup C$ = {3, 4, 7, 8}

$\Rightarrow$ $A \times (B \cup C) = \{2, 5\} \times \{3, 4, 7, 8\}$

= {(2, 3), (2, 4), (2, 7), (2, 8), (5, 3), (5, 4), (5, 7), (5, 8)}

and $(A \times B) \cup (A \times C) =$ {(2, 3), (2, 4), (2, 7), (2, 8), (5, 3), (5, 4), (5, 7), (5, 8)}

= $A \times (B \cup C)$

Important formulae/points

- *The Cartesian product is denoted A × B* Ĝ $A \times B = \{(a, b);\ a \in A,\ b \in B\}$.
- *The elements of A × B are also called 2-tuples.*
- *If A = ¤ or B = ¤ i.e. if at least one of A and B is an empty set, then A × B = ¤.*
- *A × B ÷ ¤* ę *A ÷ ¤ and B ÷ ¤*
- *A × B may or may not be equal to B × A.*
- *A × B = B × A* if and only if *A = B.*

PRACTICE PROBLEMS

PP13. Let *A* = {1, 2, 3} and *B* = {4, 5}. Find *A* × *B* and *B* × *A* and show that $A \times B \neq B \times A$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?

Solution:

Let A = set of people who like coffee

B = set of people who like tea

Given, $n(A \cup B) = 70$, $n(A) = 37$, $n(B) = 52$

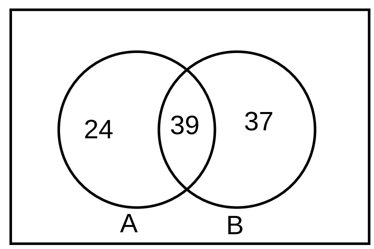
To find $n(A \cap B)$

$$n(A \cap B) = n(A) + n(B) - n(A \cup B) = 37 + 52 - 70 = 19$$

***Example* 2:**

If 53% of persons like oranges where 66% like apples, what can be said about the percentage of persons who like both oranges and apples?

Solution: Let the total number of persons = 100 $\Rightarrow$ $n(A \cup B) = 100$



Let A = {x : x like oranges}

B = {x : x likes apples)

Then $n(A) = 53$, $n(B) = 66$

$\therefore$ $A \cap B = \{x : x \text{ likes oranges and apples both}\}$

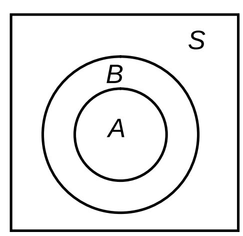
Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore$ $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 53 + 66 - 100 = 19$

***Example* 3:**

Let *A* has 3 elements and *B* has 6 elements. What can minimum number of elements in *A* Ć *B* ?

Solution: Clearly $A \cup B$ will contain minimum number of elements if $A \subseteq B$ or $B \subseteq A$



But $n(A) = 3 < 6 = n(B)$

$\therefore$ $B \not\subseteq A$ $\therefore$ $A \subset B$

Thus $A \cup B = B$ $\therefore$ $n(A \cup B) = n(B) = 6$

Thus $A \cup B$ contains at least 6 elements

***Example* 4:**

In a group of 2000 people, there are 1500, who can speak Hindi and 800, who can speak Bengali. How many can speak Hindi only? How many can speak Bengali only? How many can speak both Hindi and Bengali?

Solution: Let $A = \{x : x \text{ speaks Hindi}\},\ B = \{x : x \text{ speaks Bengali}\}$

Then $A - B = \{x : x$ speaks Hindi and can not speak Bengali}

$B - A = \{x : x$ speaks Bengali and can not speak Hindi}

$A \cap B = \{x : x$ speaks Hindi and Bengali both)

Given, $n(A) = 1500,\ n(B) = 800,\ n(A \cup B) = 2000$

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 1500 + 800 - 2000 = 300$

$\therefore$ Number of people speaking Hindi and Bengali both is 300

$n(A) = n(A - B) + n(A \cap B) \Rightarrow n(A - B) = n(A) - n(A \cap B) = 1500 - 300 = 1200$

Also, $n(B - A) = n(B) - n(A \cap B) = 800 - 300 = 500$

Thus number of people speaking Hindi only = 1200

And number of people speaking Bengali only = 500

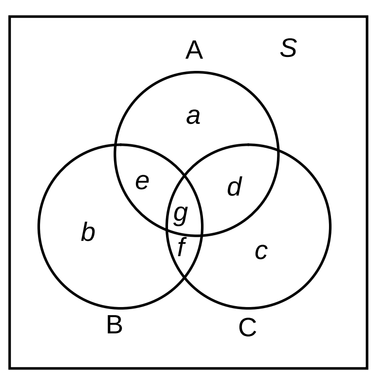
***Example* 5:**

A class has 175 students. Following is the description showing the number of students studying one or more of the following subjects in this class.

Mathematics 100, Physics 70, Chemistry 46; Physics and Chemistry 23; Mathematics and Physics 30; Mathematics and Chemistry 28; Mathematics, Physics and Chemistry 18.

How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone? Are there students who have not offered any of these three subjects.

Solution: Let *A, B* and *C* denote the sets of students studying Mathematics, Physics and Chemistry respectively.



Let us denote the number of elements (students) contained in the bounded region as shown in the diagram by *a, b, c, d, e, f* and *g* respectively.

Then $a + d + g + e = 100$

$b + f + g + e = 70$

$c + f + g + d = 46$

$g + e = 30$

$g + d = 28$

$g + f = 23$

$g = 18$

Solving these, we get

$g = 18,\ f = 5,\ d = 10,\ e = 12,\ c = 13,\ b = 35,\ a = 60$

$\therefore \quad a + b + c + d + e + f + g = 153$

So, the number of students who have not offered any these three subjects

175 – 153 = 22

Students studying Mathematics only = a = 60

Students studying Physics only = b = 35

Students studying Chemistry only = c = 13

***Example* 6:**

For any sets *A, B, C.* Using logical method, prove that

(i) $A \cap (B - A) = \phi$ **(ii)** $A - B = B - A \Leftrightarrow A = B$

(iii) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ **(iv)** $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Solution: (i) Let $x \in A \cap (B - A) \Rightarrow x \in A$ and $x \in (B - A)$

$\Rightarrow \quad x \in A$ and $(x \in B$ and $x \notin A)$

$\Rightarrow \quad (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin A)$

$\Rightarrow \quad x \in (A \cap B)$ and $x \in \phi$

$\Rightarrow \quad x \in \phi$ [$\because$ ϕ has no element]

Hence $A \cap (B - A) \subseteq \phi$ …(i)

But ϕ is a subset of each set.

$\therefore \quad \phi \subseteq A \cap (B - A)$ …(ii)

From (i) and (ii), we have, $A \cap (B - A) = \phi$

(ii) $A - B = B - A \Leftrightarrow A = B$

Only if part: Let $A - B = B - A$ …(i)

To prove $A = B$

Let $x \in A \Leftrightarrow \quad (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$

$\Leftrightarrow \quad x \in (A - B)$ or $x \in (A \cap B)$

$\Leftrightarrow \quad x \in (B-A)$ or $x \in A \cap B$ [from (i)]

$\Leftrightarrow \quad (x \in B$ and $x \notin A)$ or $(x \in B$ and $x \in A)$

$\Leftrightarrow \quad x \in B$

Hence $A = B$

If part: Let $A = B$ …(ii)

To prove $A - B = B - A$

Now, $A - B = A - A = \phi$ [$\because B = A$]

and $B - A = A - A = \phi$ [$\because B = A$]

$\therefore \quad A - B = B - A$

Thus $A = B \Rightarrow A - B = B - A$

(iii) $x \in (A \cup B) - (A \cap B)$

$\Leftrightarrow \quad x \in (A \cup B) \wedge x \notin (A \cap B)$ [$\wedge$ stands for 'and']

$\Leftrightarrow \quad (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B)$ [$\vee$ stands for 'or']

$\Leftrightarrow \quad [(x \in A \vee x \in B) \wedge (x \notin A)] \vee [(x \in A \vee x \in B) \wedge x \notin B]$

$\Leftrightarrow \quad [x \in B - A] \vee [x \in A - B]$

$\Leftrightarrow \quad x \in (B - A) \cup (A - B)$

$\Leftrightarrow \quad x \in (A - B) \cup (B - A)$ [$\because A \cup B = B \cup A$]

Thus $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

(iv) $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) or $(x \notin A$ and $x \in B)$ or $(x \in A$ or $x \in B)$

$\Leftrightarrow \quad x \in A - B$ or $x \in B - A$ or $x \in A \cap B$

$\Leftrightarrow \quad x \in (A - B) \cup (B - A) \cup (A \cap B)$

$\therefore \quad A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

EXERCISE – I

1. If $n(A)=4,\ n(B)=3,\ n(A\times B\times C)=24$ find $n(C)$.

2. If $A=\{2,\ 3,\ 5\},\ B\{2,\ 5,\ 6\}$, then find $(A-B)\times(A\cap B)$.

3. Find the value of x and y, if $(2x,\ x+y)=(6,\ 2)$.

4. If A and B are two sets such that $n(A)=17,\ n(B)=23$ and $n(A\cup B)=38$. Find $n(A\cap B)$.

5. In a group of people 50 speak both English and Hindi and 30 people speak English but not Hindi. All the people speak at least one of the two language, then find the number of people who speak English.

6. If $n(A\cup B)=60,\ n(A\cap B)=10$ and $n(A)=40$. Find $n(B)$.

7. If $n(A)=300,\ n(B)=200$ and $n(A\cup B)=400$. Find $n(A-B)$.

8. In a group of 60 persons each takes atleast one of coffee or tea. If 20 of tem take tea but not coffee and 35 take tea, then find the number of persons who take coffee but not tea.

9. If A and B are any two sets and $n(A\cup B)=n(A)+n(B)$. Find the value of $n(A\cap B)$.

10. Two finite sets have m and n elements, the number of subsets of the first set is 112 more than that of the second set, then find the value of $m,\ n$.

11. If a set A contains n distinct elements, then find the number of elements in power set A.

12. A survey of 2000 people shows that 1720 people like Maruti car and 1450 people like Hyundai car. Find the least number of people that must have liked both the cars.

13. If set $A=\{x : x\in R,\ x^2=16 \text{ and } 2x=6\}$, then write the set A in roster form.

14. Set A and set B have 3 and 6 elements respectively, then what would be the minimum number of elements in $A\cup B$.

15. If $A=\{x : x^2-5x+6=0\},\ B=\{2,\ 4\},\ C=\{4,\ 5\}$, then find $A\times(B\cap C)$.

EXERCISE – II

1. Which of the following collections are sets? Justify your answer.
(i) The collection of all most talented writers of India.
(ii) The collection of all months of a year beginning with the letter J.
(iii) The collection of handsome boys of the world.

2. Are the following sets equal?
(i) $A = \{x : x^3 - 8 = 0$ and x is a real number$\}$.
$B = \{x : x^2 + 7x - 18 = 0$ and $x > 0\}$.
(ii) A = the set of letters in the word 'ALLOY'.
B = the set of letters in the word 'LOYAL'.

3. Which of the following sets are infinite sets?
(i) The set of all circles passing through the origin.
(ii) The set of prime numbers less than 99.

4. If $A = \{2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{4, 5, 6, 7, 8\}$. Then find
(i) $(A \cup B) \cap (A \cup C)$ (ii) $A - (B \cup C)$
(iii) $A - (B - C)$ (iv) $(A \cap B) \cup (A \cap C)$

5. For sets A and B, prove the following using the properties of sets:
(i) $(A \cup B) - A = B - A$ (ii) $A \cup (B - A) = A \cup B$
(iii) $A - (A - B) = B \Leftrightarrow B \subseteq A$ (iv) $A \cup (A \cap B) = A$
(v) $(A \cap B) \cup (A - B) = A$ (vi) $(A \cup B) \cap (A \cup B') = A$

6. For sets A, B and C, prove the following using the properties of sets:
(i) $A - (B - C) = (A - B) \cup (A \cap C)$ (ii) $(A - B) - C = A - (B \cup C)$
(iii) $A \cap (A \cup B) = A$

7. For any three sets A, B, C, prove the following (by using different laws on operations of sets):
(i) $(A - B) \cup A = A$ (ii) $(A - B) \cap (B - A) = \phi$
(iii) $A - (A - B) = A \cap B$ (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$

8. If A and B are two non-empty sets having n elements in common, then show that $A \times B$ and $B \times A$ have n^2 elements in common.

9. Use logical method to prove the following:
(i) For any set A, prove that $(A')' = A$.
(ii) For any two sets A and B, prove that $A \subseteq B \Leftrightarrow B' \subseteq A'$.
(iii) For any two set A and B, prove that $A - B = \phi$ iff $A \subseteq B$.

10. In a group of 1000 people, 750 can speak Hindi and 400 can speak Bengali. All the people speak at least one of the two languages. How many can speak Hindi only? How many can speak Bengali only? How many can speak both Hindi and Bengali?

11. A college awarded 38 medals in Football, 15 in Basketball and 20 in Cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?

12. In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry and 3 had taken all three subjects. Find the number of students that had taken (i) only chemistry (ii) only mathematics (iii) only physics (iv) physics and chemistry but not mathematics (v) mathematics and physics but not chemistry (vi) only one of the subjects (vii) at least one of three subjects (viii) none of three subjects.

13. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Out of the total, 64 played both basketball and hockey, 80 played cricket and basketball and 40 played cricket and hockey, 24 played all the three games. Find the number of boys who did not play any game.

14. Shade the following sets:
(i) $A' \cap B'$ (ii) $A' \cup B'$

15. Shade the following sets:
(i) $(A \cup B) \cap (A \cup C)$ (ii) $(A \cap B) \cup (A \cap C)$

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. (i) and (ii)

PP2 . (i) $A = \{n^2 : n \in N\}$ (ii) $A = \{x : x = 4n + 1, n \in N\}$

(iii) $A = \{x : x = 7n, n \in N, 2 \le n \le 14\}$ (iv) $A = \{x : x = 3n, n \in N\}$

PP3. (i) $A = \{6, 7, 8, 9, \ldots\ldots, 26\}$ (ii) $B = \left\{\frac{-1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}\right\}$

(iii) $C = \{1, 3, 5, 7, \ldots\ldots, 19\}$ (iv) $D = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\right\}$

PP4. (i) $\leftrightarrow$ (b); (ii) $\leftrightarrow$ (c); (iii) $\leftrightarrow$ (d); (iv) $\leftrightarrow$ (a)

PP5. *A* is a null set, *B* and *C* are singleton sets

PP6. Number of subsets = $2^{n(A)} = 8$

PP7. (i) False, because {2, 3} is an element of *A* and is not its subsets.

(ii) True

(iii) False, because $2, 4 \in A$ so $\{2, 4\} \subset A$.

(iv) True, because the set on L.H.S. has the element {2, 3} which belongs to *A*.

(v) False, because $3 \notin A$.

PP11. 39%

PP12. 4 and 13

EXERCISE – I

1. 2

2. {(3, 2), (3, 5)}

3. $x = y = -1$

4. 2

5. 80

6. 30

7. 200

8. 25

9. zero

10. 4, 7

11. 2^n elements

12. 1170

13. ϕ

14. 6

15. $\{(2, 4), (3, 4)\}$

EXERCISE – II

1. (ii)

2. (i) $A = B$

(ii) $A = B$

3. (i) is infinite set

4. (i) {2, 3, 4, 5, 6, 7}

(ii) {2}

(iii) {2, 4, 5, 6}

(iv) {3, 4, 5, 6}

9. $x = 3$, $y = -1$

10. The number of people speaking Hindi only = 600, Bengali only = 250, both Hindi and Bengali = 150

11. Number of people who got medal in exactly two of the three sports = 9

12. (i) 5

(ii) 4

(iii) 2

(iv) 1

(v) 6

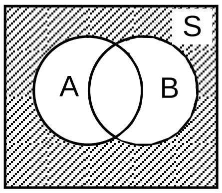
(vi) 11

(viii) 23

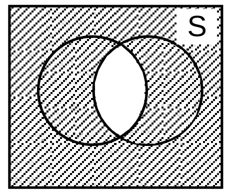
(viii) 2

13. 160

14. (i) (ii)

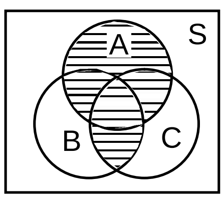


(i) $A' \cap B'$

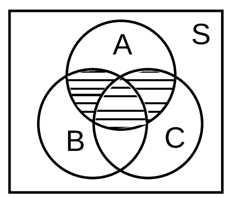


(ii) $A' \cup B'$

15 (i) (ii)



(iii) $(A \cup B) \cap (A \cup C)$



(iv) $(A \cap B) \cup (A \cap C)$