LESSON 12

PROBABILITY

1. INTRODUCTION

Probability is defined as a measure of uncertainty of events in a random experiment.

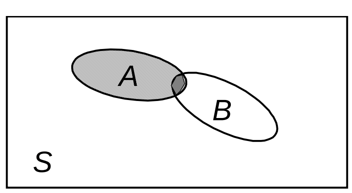
It may also be treated as a function of outcomes of the random experiment. We are also aware with axiomatic theory and classical theory of probability.

Now we will discuss about conditional probability Baye's theorem and Probability distributions like binomial.

2. CONDITIONAL PROBABILITY

Often it is required to find the probability of an event *B* under the condition that an event A occurs. This probability is called the conditional probability of *B* given A and is denoted by *P*(*B*/A). In this case A serves as a new (reduced) sample space, and the probability is the fraction of that part of set A which corresponds to A $\cap$ *B*. Thus

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{where } P(A) \neq 0)$$



The shaded portion shows the favourable region
and lined portion shows the reduced sample space.

Similarly, the conditional probability of A given B is

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

From the above two expressions, we can state the probability of intersection of two events A and B where $P(A) \neq 0$ and $P(B) \neq 0$ as $P(A \cap B) = P(A) \,.\, P\left(\frac{A}{B}\right)$ or $P(B)P\left(\frac{A}{B}\right)$.

(Multiplication theorem)

***Illustration* 1**

Question: **In producing screws, let *A* mean "screw too slim" and *B* "screw too short."**

Let *P*(*A*) = 0.1 and let the conditional probability that a slim screw is also too short be *P*(*B*/*A*) = 0.2. What is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Solution: We require the probability of occurrence of both the events together, which can be given as

$P(A \cap B) = P(A)\, P(B/A) = 0.1 \,.\, 0.2 = 0.02 = 2\%$.

3. INDEPENDENT EVENTS

If events A and B are such that

$P(A \cap B) = P(A)\, P(B)$,

they are called independent events. Assuming $P(A) \neq 0$, $P(B) \neq 0$, in this case

$P(A/B) = P(A)$, $\quad P(B/A) = P(B)$

This means that the probability of A does not depend on the occurrence or nonoccurrence of B, and conversely. This justifies the term independent.

Similarly , m events A_1 , ,…….. , A_m are called independent if

$P(A_1 \cap \ldots \cap A_m) = P(A_1) \ldots\ldots P(A_m)$

as well as for every k different events $A_{j_1}, A_{j_2}, \ldots\ldots, A_{j_k}$

$$P(A_{j_1} \cap A_{j_2} \cap \ldots\ldots \cap A_{j_k}) = P(A_{j_1})\, P(A_{j_2}) \ldots\ldots P(A_{j_k})$$

where $k = 2, 3, \ldots, m - 1$.

Accordingly, three events A, B, C are independent if

$P(A \cap B) = P(A)\, P(B)$,

$P(B \cap C) = P(B)\, P(C)$,

$P(C \cap A) = P(C)\, P(A)$

$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$

***Illustration* 2**

Question: **A and B are two independent witnesses in a case. The probability that A will speak the truth is 3/4; and that of B is 4/5. In what percentage of cases are they likely to contradict each other in stating the same fact?**

Solution: If E is the event of their contradicting each other then $E = (A \cap \overline{B}) \cup (\overline{A} \cap B)$ also $(A \cap \overline{B})$ and $(\overline{A} \cap B)$ are two mutually exclusive events.

$$P(E) = P(A \cap \overline{B}) + P\,(\overline{A} \cap B) \quad = \frac{3}{4}.\frac{1}{5} + \frac{4}{5}.\frac{1}{4} = \frac{7}{20}$$

$\therefore$ in 35% of the cases they are likely to contradict each other.

***Illustration* 3**

Question: **The independent probabilities that, A, B and C solve a Mathematical problem are $\frac{1}{3}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that just two of them only solve the problem.**

Solution: Given that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$ and A, B, C are independent events.

The problem gets solved by any two of them solving but the third one fails.

Required probability

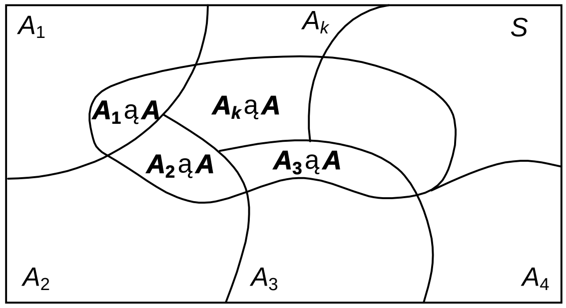
$$= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$$

$$= \frac{1}{3}.\frac{1}{3}.\frac{3}{4} + \frac{1}{3}.\frac{2}{3}.\frac{1}{4} + \frac{2}{3}.\frac{1}{3}.\frac{1}{4} \quad = \frac{3}{36} + \frac{2}{36} + \frac{2}{36} = \frac{7}{36}$$

4. TOTAL PROBABILITY

Consider a sample space S, let A_i, i = 1 to n be the set of n mutually exclusive and exhaustive set of sample space S.

Thus $A_i \cap A_j = \phi$ for $1 \le i < j \le n$ and $\sum_{i=1} P(A_i) = 1$ as $\bigcup_{i=1}^{n} A_i = S$



Let A be any event of S. Then total probability of the event A is given by

$$P(A) = \sum_{i=1}^{n} P(A_i)P(A/A_i)$$

where $P(A/A_i)$ gives us the contribution of A_i in the occurrence of A.

This result is obtained as

$$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \ldots\ldots \cup (A_n \cap A)$$

$\Rightarrow$ $P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \ldots\ldots + P(A_n \cap A)$

$$= P(A_1)P\left(\frac{A}{A_1}\right) + P(A_2)\,P\left(\frac{A}{A_2}\right) + \ldots.. + P(A_n)\,P\left(\frac{A}{A_n}\right) = \sum_{i=1}^{n} P(A_i)\,P\left(\frac{A}{A_i}\right)$$

***Illustration* 4**

Question: **An unbiased coin is tossed. If the result is head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, … 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?**

Solution: Let E_1 be the event "toss results in a head",

E_2 be the event "toss results in a tail"

A be the event "the noted number is 7 or 8"

We have $P\,(E_1) = P\,(E_2) = \frac{1}{2}$

Also $P\,(A/E_1) = P\,(7) + P\,(8) = \frac{6}{36} + \frac{5}{36} = \frac{11}{36}$

Since $7 = \{1+6,\ 2+5,\ 3+4,\ 4+3,\ 5+2,\ 6+1\}$ and $8 = \{2+6,\ 3+5,\ 4+4,\ 5+3,\ 6+2\}$

$$P\,(A/E_2) = \frac{2}{11}$$

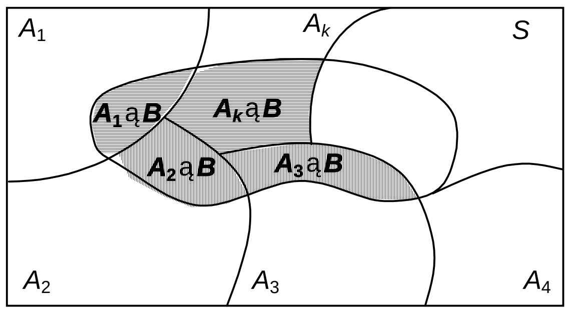
We know that $P\,(A) = P\,(A/E_1) \times P\,(E_1) + P\,(A/E_2) \times P\,(E_2) = \frac{11}{36} \times \frac{1}{2} + \frac{2}{11} \times \frac{1}{2} = \frac{193}{792}$

5. BAYES THEOREM OR INVERSE PROBABILITY

Bayes theorem gives probability of occurrence of an event when the outcome of experiment is known.

Consider a sample space *S*, let A_i, i = 1 to *n* be the set of *n* mutually exclusive and exhaustive set of sample space *S*.

Thus $A_i \cap A_j = \phi$ for $1 \le i < j \le n$ and $\sum_{i=1} P(A_i) = 1$ as $\bigcup_{i=1}^{n} A_i = S$



Let *B* be an event of *S* which has already occurred then conditional probability of occurrence of any one of the event say A_k out of the A_i , i = 1, 2, ….. *n* events is

$$P\left(\frac{A_k}{B}\right) = \frac{P(A_k \cap B)}{P(B)} = \frac{P(A_k)\,P(B/A_k)}{P(B)}$$

Now using the concept of total probability we get Baye's theorem as follows:

$$P\left(\frac{A_k}{B}\right) = \frac{P(A_k)\,P(B/A_k)}{\sum_{i=1}^{n} P(A_i)\,P(B/A_i)}$$

Illustration 5

Question: **In a factory manufacturing bolts; machines *A*, *B* and *C* manufacture respectively 20%, 30% and 50% of the total production. Of their outputs 2%, 3% and 5% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine *C*?**

Solution: Let *A*, *B*, *C* be the events that a bolt selected at random is manufactured respectively by machines *A*, *B* and *C* respectively so that $P(A) = 2/10$; $P(B) = 3/10$ and $P(C) = 5/10$

Let *D* be the event that the bolt selected at random is defective, so that

$P(D/A) = 2/100;$

$P(D/B) = 3/100$

and $P(D/C) = 5/100;$

$$P(C/D) = \frac{P(D/C)\,.\,P(C)}{P(D/A)\,.\,P(A) + P(D/B)\,.\,P(B) + P(D/C)\,.\,P(C)}$$

$$= \frac{5 \times 5}{4 + 9 + 25} = \frac{25}{38}$$

Illustration 6

Question: **In a context A targets *B* and both *B* and *C* target *A*. The probability of *A, B, C* hitting their targets are $\frac{2}{3}$ and $\frac{1}{3}$ respectively. They shoot simultaneously and *A* is hit. Find the probability that *B* hits his target where as *C* does not.**

Solution: Let E = the event that A is hit

E_1 = the event that B hits A

E_2 = the event that C hits A

Given $P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{3}$

Let $E_1 \cap E_2' = A_1;\ E_1 \cap E_2 = A_2;\ E_2 \cap E_1' = A_3$

Clearly $E = A_1 \cup A_2 \cup A_3$

By Bayes's theorem

Required probability = $P(A_1/E) = \dfrac{P(A_1)P(E/A_1)}{P(A_1).P(E/A_1) + P(A_2).P(E/A_2) + P(A_3).P(E/A_3)}$

$$= \frac{P(A_1)}{P(A_1) + P(A_2) + P(A_3)} \qquad [\because\ A_1,\ A_2,\ A_3 \subseteq E,\ \therefore\ P(E/A_1) = P(E/A_2) = P(E/A_3) = 1]$$

$$= \frac{P(E_1).P(E_2)}{P(E_1).P(E_2') + P(E_1).P(E_2) + P(E_2).P(E_1')} = \frac{\frac{1}{2}.\frac{2}{3}}{\frac{1}{2}.\frac{2}{3} + \frac{1}{2}.\frac{1}{3} + \frac{1}{3}.\frac{1}{2}} = \frac{1}{2}$$

6. RANDOM VARIABLE AND PROBABILITY DISTRIBUTION

A random variable is generally described as a variable whose values are the result of some changing conditions. Consider a simultaneous throw of two coins. The sample space is

$S = \{HH, HT, TH, TT\}$

Let X denote the number of heads in a point of the sample space S. Then

$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$.

Thus X takes values 0, 1, 2 only and no more. Here we say that X is a random variable or a stochastic variable. We have the following definition.

A random variable is a real valued function *X* defined over the sample space of an experiment i.e. a random variable is a function which associates to each pont of a sample space, a unique real number.

- Random variables are denoted by *X, Y, Z*.
- More than one random variables can be defined on the same sample space.

e.g. Let Y denote the number of heads minus the number of tails for each outcome of the sample space S.

Then $Y(HH) = 2 - 0 = 2$

$Y(HT) = 1 - 1 = 0$

$Y(TH) = 1 - 1 = 0$

$Y(TT) = 0 - 2 = -2$

Thus X and Y are two different random variables defined on the same sample space S.

Illustration 7

Question: **A bag contains 2 white and 1 red ball. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If *X* denotes the number of red balls recorded in the two draws, describe *X*.**

Solution: Let the balls in the bag be denoted by W_1, W_2, R

$S = \{W_1W_2, W_1W_2, W_2W_1, W_2W_2, W_1R, W_2R, RW_1, RW_2, RR\}$

Now for $W \in S$

$X(W) =$ number of red balls

Therefore $X(\{W_1W_1\}) = X(\{W_1W_2\}) = X(\{W_2W_1\}) = X(\{W_2W_2\}) = 0$

$X(\{W_1R\}) = X(\{W_2R\}) = X(\{RW_1\}) = X(\{RW_2\}) = 1$

and $X(\{RR\}) = 2$

$X(\{W, R\}) = X(\{W_2R\}) = X(\{RW_1\}) = X(\{RW_2\}) = 1$

and $X(\{RR\}) = 2$

Probability distribution of random variable

A distribution, in which values of the random variable and their corresponding probabilities are given is called the probability distribution of the random variable.

Let us suppose that a discrete variable X assumes values $x_1, x_2, ...x_n$ with probability $p_1, p_2 .....p_n$ respectively, where $p_1 + p_2 + ... + p_n = 1$ and $0 \le p_i \le 1$ for all $i = 1, 2, ...n$.

Then the following table describes the probability distribution.

X	x_1	x_2	x_3	x_4	………	x_n
$P(X)$	p_1	p_2	p_3	p_4	………	p_n

Example:

Let X be the random variable denoting the number of tails in a simultaneous throw of two coins. Then clearly X can take the values 0, 1, 2.

$X(TT)=2,\ X(HT)=1,\ X(TH)=1$ and $X(HH)=0$

Let $P(X)=$ Prob. of the variable X.

Then $P(X=0)=P\text{ (no tail)}=\frac{1}{4}$

$P(X=1)=P\text{ (one tail) }=\frac{2}{4}=\frac{1}{2}$

$P(X=2)=P\text{ (two tails) }=\frac{1}{4}$

Note:

$$P(X=0)+P(X=1)+P(X=2)=1\ \left[\because \frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1\right]$$

We can write the above result in the following form:

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Clearly, each of the probability is a non-negative fraction (never greater than 1) and their sum is 1.
The above form is the probability distribution of the random variable X.

Illustration 8

Question: **An urn contains 5 white and 3 red balls. Find the probability distribution of the number of read balls, with replacements, in three draws.**

Solution: Let R be the event of drawing a red ball.

Let X denote the discrete random variable "no. of red balls" in a drawn of three balls. Then

$X=0,1,2,3.$ Here $P(R)=\frac{3}{8}$ and $P(\overline{R})=\frac{5}{8}.$

$$P(X=0)=P(\overline{R}_1\overline{R}_2\overline{R}_3)=P(\overline{R}_1)P(\overline{R}_2)P(\overline{R}_3)=\frac{5}{8}\times\frac{5}{8}\times\frac{5}{8}=\frac{125}{512}$$

$$P(X=1)=P(\overline{R}_1R_2R_3)+P(R_1\overline{R}_2R_3)+P(R_1R_2\overline{R}_3)=3\left(\frac{5}{8}\times\frac{3}{8}\times\frac{3}{8}\right)=\frac{135}{512}$$

$$P(X=3)=P(R_1R_2R_3)=\frac{3}{8}\times\frac{3}{8}\times\frac{3}{8}=\frac{27}{512}.$$

Hence the required probability dist. is

$X:$	0	1	2	3
$P(X)$"	$\frac{125}{512}$	$\frac{225}{512}$	$\frac{135}{512}$	$\frac{27}{512}$

7. BINOMIAL DISTRIBUTION

In n independent trials of a random experiment, let X be the number of times an event A occurs. In each trial, event A has same probability as $P(A) = p$ referred to as success. Then in a trial non-occurrence of A is referred as failure and given by $q = 1 - p$.

Here X can assume values from o to n. Now $X = r$ means A occurs in r trials and $(n - r)$ it does not occur. this may look as

$$\underbrace{A\ A\ A.........A}_{r\text{ times}}\quad\underbrace{\overline{A}\ \overline{A}.........\overline{A}}_{n-r\text{ times}}$$

here $\overline{A}$ means complement of A. Using the assumption that trials are independent, that is, they do not influence each other, hence has the probability

$$\underbrace{p\ p\ ....p}_{r\text{ times}}\quad\underbrace{q\ q.........q}_{n-r\text{ times}}=p^r\ q^{n-r}$$

and it can be arranged in $\frac{n!}{r!(n-r)!}={}^nC_r$ ways.

Hence the probability of getting r successes or occurrence of A in r trials out of n independent trials is

$$\boldsymbol{P(X=r)={}^nC_r\,p^r\,q^{n-r}}$$

which is denoted as binomial distribution of random variable X.

- The probability of getting atleast k successes in

$$\boldsymbol{P(X\geq k)=\sum_{r=k}^{n}{}^nC_r\,p^r q^{n-r}}$$

- The probability of getting almost K successes is

$$\boldsymbol{P(X\leq k)=\sum_{r=0}^{k}{}^nC_r\,p^r q^{n-r}}$$

- $\boldsymbol{\sum_{r=0}^{n}{}^nC_r\,p^r q^{n-r}=(p+q)^n=1}$

Illustration 9

Question: **Ten coins are tossed simultaneously. Find the probability of getting at least 7 heads.**

Solution: In this case $n = 10$; $p = 1/2$; $q = 1/2$

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= \left({}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + 1\right)\frac{1}{2^{10}} = \frac{176}{2^{10}}$$

Illustration 10

Question: **Numbers are chosen at random, one at a time, from the two digit numbers 00, 01, 02, … 99 with replacement. An event *E* occurs if and only if the product of the two digits of a selected number is 18. If four numbers are selected, find the probability that *E* occurs at least 3 times.**

Solution: The numbers, whose two digits product is 18, are 29, 36, 63, 92

$$P(E) = \frac{4}{100} = 0.04$$

Since there is replacement, $P(E)$ remains the same for every selection. Four such selections are made.

Probability that E occurs, at least thrice

= Probability that E occurs thrice and fails to occur once + Probability that E occurs all the four times

$$= {}^4C_3\ (0.04)^3\ (0.96) + (0.04)^4 = 0.00024832$$

Recurrence Formula for Binomial Distribution

We know that $P(r) = {}^nC_r q^{n-r} p^r$ [**note:** $P(r)$ means $P(X = r)$]

$$P(r+1) = {}^nC_{r+1} q^{n-r-1} p^{r+1}$$

$$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\cdot\frac{q^{n-r-1}}{q^{n-r}}\cdot\frac{p^{r+1}}{p^r}$$

$$= \frac{n-r}{r+1}\cdot\frac{p}{q}$$

Hence $P(r+1) = \frac{n-r}{r+1}\cdot\frac{p}{q}\ P(r)$,

which is the required recurrence formula for binomial distribution.

- If $P(0)$ is known, then we find $P(1)$, $P(2)$, …… with the help of recurrence formula.
- If $p = \frac{1}{2}$, then $q = \frac{1}{2}$. Then the binomial distribution is called symmetrical binomial distribution.

Mean and Variance of binomial distribution

If $x_1, x_2, ...x_n$ are the values of a random variable X and $p_1, p_2, ...p_n$ are the corresponding probabilities, then mean (μ) and variance (σ^2) of the probability distribution are given by

$$\mu = \Sigma p_i x_i$$

$$= 0.p(0) + 1.p(1) + .2p(2) + ... + n.p(n)$$

$$= 1.\ {}^nC_1 pq^{n-1} + 2.\ {}^nC_2 p^2 q^{n-2} + ... + n\ {}^nC_n p^n$$

$$= npq^{n-1} + 2\frac{n(n-1)}{2}.p^2q^{n-2} + 3.\frac{n(n-1)(n-2)}{1.2.3}.p^3q^{n-3} + ... + n.p^n$$

$$= np[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{1.2}p^2q^{n-3} + ... + p^{n-1}]$$

$$= np[\ {}^{n-1}C_0 q^{n-1} + C_1 pq^{n-2} + {}^{n-1}C_2 p^2 q^{n-3} + ... + {}^{n-1}C_{n-1}p^{n-1}]$$

$$= np(q+p)^{n-1} = np(1)^{n-1} \qquad [\because q + p = 1]$$

$$= np$$

Hence the mean of the distribution i.e. $\mu = np$

Again variance σ^2 is given by

$$\sigma^2 = \Sigma p_i x_i^2 - \mu^2$$

$$= 0^2 p(0) + 1^2.p(1) + 2^2.p(2) + ... + n^2.p(n) - \mu^2$$

$$= 1.\ {}^nC_1 pq^{n-1} + 2^2.\ {}^nC_2 p^2 q^{n-2} + ... + n^2.\ {}^nC_n p^n - \mu^2$$

$$= [npq^{n-1} + 4\frac{n(n-1)}{1.2}p^2q^{n-2} + 9.\frac{(n-1)(n-2)}{1.2.3}p^3q^{n-3} + ... + n^2.p^n] - \mu^2$$

$$= np[q^{n-1} + 2.(n-1)pq^{n-2} + 3.\frac{(n-1)(n-2)p^2}{1.2}q^{n-3} + ... + np^{n-1}] - \mu^2$$

$$= np[\ {}^{n-1}C_0 q^{n-1} + 2^{(n-1)}C_1 p^1 q^{n-2} + 3.\ {}^{(n-1)}C_2 p^2 q^{n-3} + ... + np^{n-1}] - \mu^2$$

$$= np[(\ {}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 pq^{n-2} + {}^{n-1}C_2 p^2 q^{n-3} + ... + {}^{n-1}C_{n-1}p^{n-1})]$$

$$+ (\ {}^{n-1}C_1 pq^{n-2} + 2.\ {}^{n-1}C_2 p^2 q^{n-3} + ... + (n-1.\ {}^{n-1}C_{n-1}p^{n-1})] - \mu^2$$

$$= np[(q+p)^{n-1} + (n-1)p(\ {}^{n-2}C_0.q^{n-2}) + {}^{n-2}C_1 pq^{n-3} + ... + {}^{n-2}C_{n-2}p^{n-2})] - \mu^2$$

$$\left[\because r.\frac{{}^{n-1}C_r}{n-r} = {}^{n-1}C_{r-1}\ for\ r = 1, 2, 3....n-1\right]$$

$$= np[(1)^{n-1} + (n-1)p.(q+p)^{n-2}] - n^2p^2 \qquad [\because \mu = np]$$

$$= np[1 + (n-1)p.(1)^{n-2}] - n^2p^2$$

$$= np[1 + (n-1)p] - n^2p^2 = np[1 + np - p] - n^2p^2$$

$$= np[np + q] - n^2p^2 \qquad [\because 1 - p = q]$$

$= n^2p^2 + npq - n^2p^2 = npq$

Hence Variance of Binomial Distribution is given by $\sigma^2 = npq$

- Standard Deviation (S.D.) for the Binomial Distribution $= \sigma = \sqrt{npq}$
- Variance of Binomial Distribution is less than its mean

$\because$ Variance $= npq \ \leq np$

Mean = np

Hence Variance $\leq$ Mean of the Binomial Distribution.

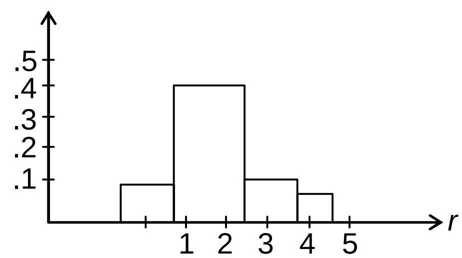
i.e. mean of the binomial distribution is always greater than the variance.

Illustration 11

Question: **Calculate $P(r)$ for r = 1, 2, 3, 4 and 5 by using the recurrence formula of the binomial distribution; use p Z $\frac{1}{3}$ and n Z 5. Hence, draw the histogram for the distribution.**

Solution: We know that

$$\Leftrightarrow P(r+1) = \frac{n-r}{r+1}.\frac{p}{q}.P(r) \qquad \ldots(1)$$



[Recurrence Formula for Binomial Distribution]

Putting $p = \frac{1}{3}, q = 1 - \frac{1}{3} = \frac{2}{3}$ and $n = 5,$ we get

$$P(r+1) = \frac{5-r}{2(r+1)}P(r), r = 0, 1, 2, 3, 4.$$

Now r = 0, 1, 2, 3, 4, in (1), we get

$$P(1) = \frac{5-0}{2(0+1)}P(0) = \frac{5}{2}(.13) = .33 \ ; \quad P(2) = \frac{5-1}{2(1+1)}P(1) = .33$$

$$P(3) = \frac{5-2}{2(3)}P(2) = \frac{1}{2}(.33) = .16 \ ; \quad P(4) = \frac{5-3}{2(3+1)}P(3) = \frac{1}{4}(.16) = .04$$

$$P(5) = \frac{5-4}{2(4+1)}P(4) = \frac{1}{10}(.04) = .004 = 0$$

Correct upto two decimal places.

PRACTICE PROBLEMS

PP1. Determine the probability of drawing 4 white balls and 2 balls without replacement, from a bag containing 1 Red, 4 Black, and 6 White balls.

PP2. There are 10 tickets in a lottery. 5 wins and 5 losses. 2 tickets are taken. What is the probability of a win?

PP3. Two guns fire simultaneously at the same target. The probability of a hit from the first is 0.7 and from the second one is 0.6. What is the probability that the target is hit?

PP4. A problem in Mathematics is given to three students *A*, *B* and *C*. Their respective chances of solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability that the problem is solved?

PP5. If $\frac{1+3p}{3}, \frac{1-p}{4}$, and $\frac{1+2p}{2}$ are the probabilities of three mutually exclusive events, show that $\frac{1}{3} \le p \le \frac{1}{2}$

PP6. If *k* objects are distributed at random among *k* persons. Find the probability that at least one of them will not get anything.

PP7. One bag contains 3 white and 2 black balls. A second bag contains 5 white and 3 black balls. If a bag is chosen at random and a ball is taken from it, what is the chance that it is white?

PP8. One bag contains 3 white and 2 black balls. A second bag contains 5 white and 3 black balls. If a bag is chosen at random and a ball is taken from it and it is found to be a white ball, what is the chance that it is taken from the first bag?

PP9. 16 coins are tossed simultaneously. What is the chance of getting at least 14 heads?

PP10. A die is thrown 6 times; getting the face 6 up is considered a success. What is the probability that there are at least 3 successes?

PP11. If six throws are made with a pair of dice, what is the chance of throwing doublets at least four times?

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

A sportsman's chance of shooting an animal at a distance $r\ (\geq a)$ is $\dfrac{a^2}{r^2}$. He fires when $r = 2a$ and if he misses, he reloads and fires when $r = 3a, 4a, 5a$ …. If he misses at a distance na, the animal escapes. What are the odds against the sportsman?

Solution:

$$P\ (r) = \frac{a^2}{r^2}$$

$$P\ (2a) = \frac{1}{4},\ P\ (3a) = \frac{1}{9},\ P\ (4a) = \frac{1}{16} \cdot etc.$$

The sportsman succeeds if

(a) he hits the first time or

(b) misses the first time but succeeds at the second or

(c) misses the first and second time but succeeds in the third and so on.

$\therefore$ probability of success

$$= \frac{1}{4} + \frac{3}{4} \times \frac{1}{9} + \frac{3}{4} \times \frac{8}{9} \times \frac{1}{16} + \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{1}{25} + \text{.......}\ (n-1) \text{ terms}$$

$$p = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \text{.... upto } (n-1) \text{ terms}$$

$$2p = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \text{.......... upto } (n-1) \text{ terms}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \text{.........} + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n} = \frac{n-1}{n}$$

***Example* 2:**

Of the three independent events, the chance that only the first occurs is *a*, that only the second occurs is *b*, and that only the third occurs is *c*. Show that the probabilities of occurrence of these three events are respectively $\dfrac{a}{a+x}, \dfrac{b}{b+x}, \dfrac{c}{c+x}$ where *x* is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

Solution:

Let E_1, E_2, E_3 be three independent events and E_1', E_2', E_3' be their complements. Then

$$P\ (E_1 \cap E_2' \cap E_3') = P\ (E_1) \cdot P\ (E_2') \cdot P\ (E_3') = a \qquad \text{… (i)}$$

since E_1, E_2, E_3 are independent

$$P\ (E_1' \cap E_2 \cap E_3') = P\ (E_1')\ P\ (E_2)\ P\ (E_3') = b \qquad \text{… (ii)}$$

and $P\,(E_1' \cap E_2' \cap E_3) = P\,(E_1')\,P\,(E_2')\,P\,(E_3) = c$ … (iii)

Denote $P\,(E_1')\,P\,(E_2')\,P\,(E_3')$ *by* x

Then $\dfrac{P\,(E_1)}{P\,(E_1')} = \dfrac{a}{x}$, this implies $\dfrac{P\,(E_1)}{1 - P\,(E_1)} = \dfrac{a}{x}$ or $P\,(E_1) = \dfrac{a}{a + x}$

Similarly, we get $P\,(E_2) = \dfrac{b}{b + x}$ and $P\,(E_3) = \dfrac{c}{c + x}$ … (iv)

Multiplying (i), (ii) and (iii), we get

$$\frac{abc}{(a + x)\,(b + x)\,(c + x)}\,x^2 = abc \text{ or } (a + x)\,(b + x)\,(c + x) = x^2$$

***Example 3*:**

A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4; and the probability that it contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement until all the defective are found. What is the probability that the testing procedure stops at the twelfth testing?

Solution: Suppose A is the event that the testing procedure ends at the twelfth testing.

A_1 = {the event that the lot contains 2 defective}

A_2 = {the event that the lot contains 3 defective}

$$P(A_1) = 0.4;\, P(A_2) = 0.6\ ,\quad P(A) = P(A_1)\,P(A/A_1) + P(A_2)\,P(A/A_2)$$

$$= 0.4\left\{\frac{{}^{18}C_{10}\,.\,{}^{2}C_1}{{}^{20}C_{11}}\cdot\frac{1}{9}\right\} + 0.6\left\{\frac{{}^{17}C_9\,.\,{}^{3}C_2}{{}^{20}C_{11}}\cdot\frac{1}{9}\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10}\cdot\frac{\underline{|18}}{\underline{|10}\,\underline{|18}}\cdot\frac{\underline{|11}\,\underline{|9}}{\underline{|20}}\cdot 2 + \frac{6}{10}\cdot\frac{\underline{|17}}{\underline{|9}\,\underline{|18}}\cdot\frac{\underline{|11}\,\underline{|9}}{\underline{|20}}\cdot 3\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10}\times\frac{11\times 9}{19\times 20}\,.\,2 + \frac{6}{10}\times\frac{9\times 10\times 11}{18\,.\,19\,.\,20}\,.\,3\right\} = \frac{44}{1900} + \frac{55}{1900} = \frac{99}{1900}$$

***Example 4*:**

Find the probability distribution of number of doublets in three throws of a pair of dice.

Solution:

Let X denote the number of doublets. Possible doublets are

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

Clearly X can take the values 0, 1, 2 or 3.

Probability of getting a doublet $= \dfrac{6}{36} = \dfrac{1}{6}$

Probability of not getting a doublet $=1-\frac{1}{6}=\frac{5}{6}$

Now, $P(X=0)=P$ (no doublet) $=\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$

$P(X=1)=P$ (one doublet and two non-doublets)

$=\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=3\left(\frac{1}{6}\times\frac{5^2}{6^2}\right)=\frac{75}{216}$

$P(X=2)=P$ (two doublets and one non-doublet)

$=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=3\left(\frac{1}{6^2}\times\frac{5}{6}\right)=\frac{15}{216}$

and $P(X=3)=P$ (three doublets) $=\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{216}$

Thus, the required probability distribution is

X	0	1	2	3
$P(X)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Verification.

Sum of the probabilities $\Rightarrow \sum_{i=1}^{n} p_i=\frac{125}{216}+\frac{75}{216}+\frac{15}{216}+\frac{1}{216}=\frac{125+75+15+1}{216}=\frac{216}{216}=1$

***Example* 5:**

A set has *n* elements. A subset *P* of A is selected at random. All the elements of *P* are returned to *A*. The subset *Q* of *A* is formed. Find the probability that *P* and *Q* have no common element.

Solution: The subset *P*, formed first, may contain none, one, two, … or all elements of *A*. *P* and *Q* have no common elements. Any elements. Any element *x* of A has one of four possibilities.

(i) $x \in P$; $x \in Q$

(ii) $x \in P$; $x \notin Q$

(iii) $x \notin P$; $x \in Q$

(iv) $x \notin P$; $x \notin Q$

Case (ii), (iii), (iv) correspond to *P* and *Q* having no common element.

Hence the probability that this element *x* is not in *P* or not in *Q* is $3/4$. The same is true of the other elements. Hence the probability is ${}^nC_n\left(\frac{3}{4}\right)^n\left(\frac{1}{4}\right)^0$

***Example* 6:**

Find the binomial distribution whose mean is 9 and whose standard deviation is $\frac{3}{2}$.

Solution:

Let *p, q* be the probability of success and failure is any one trial and *n* be the number of trials.
Then the binomial distribution is $(q+p)^n$ …(1)
For binomial distribution, we have
Mean $= np = 9$ …(2)

S.D. = $\sqrt{npq} = \frac{3}{2}$ …(3)

From (2) and (3), $\sqrt{9q} - \frac{3}{2}$ or $9q = \frac{9}{4}$ $\therefore q = \frac{1}{4}$

But $p + q = 1$ $\therefore p = 1 - \frac{1}{4} = \frac{3}{4}$

Since $np = 9$ $\therefore n\left(\frac{3}{4}\right) = 9$ $\therefore n = \frac{36}{3} = 12$

Hence the required binomial distribution is $\left(\frac{1}{4} + \frac{3}{4}\right)^{12}$

***Example* 7:**

If *E* and *F* are two independent events. The probability that both *E* and *F* happens is $\frac{1}{12}$ and the probability that neither *E* nor *F* happens is $\frac{1}{2}$. Then find the value of probability of P(E) or P(*F*).

***Solution*:**

By question $P(E \cap F) = \frac{1}{12}$ and $P(\bar{E} \cap \bar{F}) = \frac{1}{2}$

But $P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F}) = 1 - P(E \cup F)$

Therefore $P(E \cup F) = 1 - P(\bar{E} \cap \bar{F}) = 1 - \frac{1}{2} = \frac{1}{2}$ i.*e*. $P(E) + P(F) - P(E \cap F) = \frac{1}{2}$

$\therefore$ $P(E) + P(F) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$ … (i)

Since *E* and F are independent events $P(E \cap F) = P(E) \times P(F)$

$\therefore$ $P(E) \cdot P(F) = \frac{1}{12}$ … (ii)

(i) and (ii) $\Rightarrow$ *either* $P(E) = \frac{1}{3}$ *and* $P(F) = \frac{1}{4}$

or $P(E) = \frac{1}{4}$ and $P(F) = \frac{1}{3}$

***Example* 8:**

A bag contains *n* + 1 coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is 7/12, then determine the value of *n*.

***Solution*:**

Let E_1, denote the event a coin with two heads is selected and E_2 denote the event a fair coin is selected. Let A be the event the toss results in heads. Then,

$$P(E_1) = \frac{1}{n+1},\ P(E_2) = \frac{n}{n+1},\ P\left(\frac{A}{E_1}\right) = 1$$

and $P\left(\frac{A}{E_2}\right) = \frac{1}{2}$

$$P(A) = P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right)$$

$\Rightarrow \quad \frac{7}{12} = \frac{1}{n+1} \times 1 + \frac{n}{n+1} \times \frac{1}{2}$ [$\because P(A) = 7/12$]

$\Rightarrow \quad 12 + 6n = 7n + 7 \Rightarrow n = 5.$

***Example* 9:**

Bag *A* contains 2 white and 3 red balls and bag *B* contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find probability that it was drawn from bag *B*.

Solution:

Let the event of drawing a red ball be denoted by L

and the event of choosing bag A be denoted by M

and the event of choosing bag B be denoted by N.

We have $P(M) = P(N) = \frac{1}{2}$

Now $P(L / M)$ = Probability of drawing a red ball from bag A = $\frac{3}{5}$ and $P(L / N) = \frac{5}{9}$.

Required probability = $P(N / L) = \frac{P(N) \cdot P(L/N)}{P(N) \cdot P(L/N) + P(M) \cdot P(L/M)}$ (Baye's formula)

$$= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{3}{5}} = \frac{25}{52}$$

Note: The required probability *p* may be easily found without any elaborate procedure.

We can write $p = \dfrac{\text{The probability of choosing bag B and taking a red ball from it}}{\text{The probability of choosing either bag and taking a red ball from it}}$

$$= \frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{5}{9}+\frac{1}{2}\cdot\frac{3}{5}} = \frac{25}{52}$$

***Example* 10:**

Suppose *A* and *B* are two equally strong table tennis players. Which of the following two events is more probable?

(a) *A* beats *B* in exactly 3 games out of 4 or (b) *A* beats *B* in exactly 5 games out of 8

Solution:

Since *A* and *B* are equally strong players, the probability that *A* beats *B* is $\frac{1}{2}$ and the probability that *A* loses to *B* is also $\frac{1}{2}$.

(a) *A* beats *B* in exactly three games out of 4.

In this case, *A* has to lose one game and win three for which the probability is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 = \frac{1}{16}$

But this one game which he loses could be any one of the four.

The four events LWWW, WLWW, WWLW and WWWL are mutually exclusive and the probability for each one is $\frac{1}{16}$.

$\therefore$ Probability of winning exactly 3 games out of $4 = 4 \times \frac{1}{16} = \frac{1}{4}$

(b) *A* beats *B* in exactly 5 games out of 8.

In this case, *A* has to lose 3 games out of 8. These events of the type LLLWWWWW can be occur in $^8C_3$ ways and the corresponding probability

$$= {}^8C_3 \times \left(\frac{1}{2}\right)^8 = \frac{7}{32}$$

Now $\frac{1}{4} = \frac{8}{32} > \frac{7}{32}$

Hence, the first event is more probable than the second.

EXERCISE – I

1. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random. The event A, B, C are defined as

A = {the first bulb is defective}

B = {the second bulb is non-defective}

C = {the two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent.

(ii) A, B, C are independent.

2. If two events A and B are such that $P(A^c) = 0.3; P(B) = 0.4$ and $P\ (A \cap B^c) = 0.5$, find $P(B/(A \cup B^c))$

3. A consignment of 15 television sets contain 4 defectives. They are taken out one by one at random and examined. The ones that are examined are not put back. What is the probability that the ninth one examined is the last defective?

4. The probability that a certain electronic component fails when first used is 0.10. If it does not fail immediately, the probability that it lasts for one year is 0.99. Show that the probability that it lasts for one year is 0.891.

5. There are two bags one of which contains 3 Black and 4 White balls while the second one contains 4 Black and 3 White balls. A die is cast. If the face 1 or 3 turns up, a ball is chosen from the first bag, otherwise the ball is chosen from the second bag. Find the probability that a Black ball is chosen.

6. A man takes one step forward with probability 0.4 and one step backward with probability 0.6. Find the probability that at the end of 11 steps he is one step away from the starting point.

7. There are three boxes B_1, B_2 and B_3. B_1 contains two gold coins. B_2 contains two silver coins and B_3 contains one gold and one silver coins. A box is chosen at random and a coin is drawn from it. If it happens to be a gold coin, what is the probability that the other coin in the box is gold?

8. Three players A, B, C toss a coin cyclically in that order that is *ABCABCAB*… till a head is got. Let p be the probability that a coin shows a head. Let α, β, γ by respectively the probabilities that A, B, C get the first head. Prove that $\beta = (1 - p)\ \alpha$. Determine α, β, γ in terms of p.

9. A slip of paper is given to a person who marks at with either a plus or a minus sign. The probability of his writing plus is $\frac{1}{3}$. A passes the slip to *B* who may either leave it alone or change the sign before passing it to *C*. Next *C* passes it on to *D*, after perhaps changing the sign. The referee sees the plus sing on the slip. If *B*, *C*, *D* change the sign with probability $\frac{2}{3}$, what is the probability that A originally wrote the plus sign?

10. A doctor claims that 60% of the patients he examines are allergic to some type of weed. What is the probability that (i) exactly 3 of his next 4 patients are allergic to weeds?

(ii) none of his next 4 patients is allergic to weeds?

EXERCISE – II

1. Find the probability distributions of the number of heads when three coins are tossed.

2. Six coins are tossed simultaneously. Find the probability of getting

(i) 3 heads (ii) no heads (iii) at least one head

3. A box contain 13 bulbs, out of which 5 are defective. 3 bulbs are randomly drawn, one by one without replacement, from the box. Find the probability distribution of the number of defective bulbs.

4. A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant Ii, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?

5. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and a truck is 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?

6. There are three urns *A, B* and *C*. Urn A contains 4 white balls and 5 blue balls. Urn B contains 4 white balls and 3 blue balls. Urn C contains 2 white balls and 4 blue balls. One ball is drawn from each of these urns. What is the probability that out of these three balls drawn, two are white balls and one is a blue ball?

7. Three groups of children contain 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys respectively. One child is selected at random from each group. Find the chance that the three selected comprise one girl and 2 boys.

8. Two persons *A* and *B* throw a die alternately till one of them gets a ‘three’ and wins the game. Find their respectively probabilities of winning, if *A* begins.

9. A cord from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.

10. Three persons *A, B, C* throw a die in succession till one gets a ‘six’ and wins the game. Find their respective probability of winning, if *A* begins.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\frac{15}{77}$

PP2. $\frac{7}{9}$

PP3. 0.88

PP4. $\frac{3}{4}$

PP6. $1-\frac{\underline{|k}-1}{k^k-1}$

PP7. 5/36

PP8. $\frac{24}{49}$

PP9. $\frac{137}{2^{16}}$

PP10. $\frac{2906}{6^6}$

PP11. $\frac{406}{6^6}$

ANSWERS TO EXERCISE – I

1. (i) A, B, C are pairwise independent (ii) A, B, C are not independent

2. 0.25

3. $\frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} \times \frac{1}{7}$

4. 0.891

5. $\frac{11}{21}$

6. $462\ (0.24)^5$

7. $\frac{2}{3}$

8. $\alpha = \frac{p}{1-(1-p)^3}$, $\beta = \frac{p(1-p)}{1-(1-p)^3}$, $\gamma = (1-p)^2 - \frac{p}{1-(1-p)^3}$,

9. $\frac{13}{41}$

10. $\frac{16}{625}$

ANSWERS TO EXERCISE – II

1.

X:	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

2. (i) 5/6 (ii) 1/64 (iii) 63/64

3.

X:	0	1	2	3
P(X)	28/143	70/143	40/143	5/143

4. $\frac{27}{83}$

5. $\frac{1}{6}$

6. $\frac{64}{189}$

7. $\frac{13}{32}$

8. $\frac{5}{11}$

9. $\frac{11}{50}$

10. $\frac{25}{91}$