LESSON 11

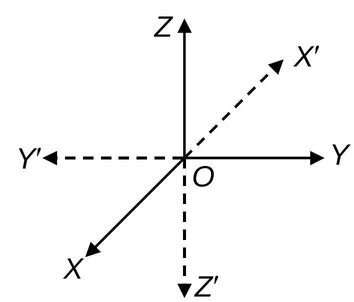
THREE DIMENSIONAL GEOMETRY

1. INTRODUCTION

Just like a point is specified in two-dimensions with respect to two perpendicular lines, a point in three-dimensional space is specified with respect to three mutually perpendicular lines *OX*, *OY* and *OZ* called as reference or coordinate axes. Every point in space is uniquely expressed as *P*(*x*, *y*, *z*).

The coordinate system normally used is called ‘the right-handed rectangular Cartesian coordinate system.

The planes containing the lines X′OX, Y′OY and Z′OZ in pairs, determine three mutually perpendicular planes XOY, YOZ and ZOX.



2. DISTANCE AND SECTION FORMULA

The distance between two points *P* and *Q* having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Distance of a point *P* (*x*, *y*, *z*) from origin is given by $\sqrt{x^2 + y^2 + z^2}$.

The coordinates of a point *R* which divides *PQ* internally in the ratio *m* : *n* are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Similarly if *R* divides *PQ* externally in the ratio of *m* : *n*, then the coordinates of *R* are

$$\left(\frac{mx_2 - nx_1}{m-n},\ \frac{my_2 - ny_1}{m-n},\ \frac{mz_2 - nz_1}{m-n}\right)$$

Illustration 1

Question: **Find the distance between the points *P* (J2, 4, 1) and *Q* (1, 2, J5).**

Solution: We have $PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (-5-1)^2}$

$= \sqrt{9+4+36} = 7$ units.

Illustration 2

Question: **Prove by using distance formula that the points *P*(1, 2, 3), *Q*(J1, J1, J1) and *R*(3, 5, 7) are collinear.**

Solution: We have $PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2}$

$= \sqrt{4+9+16} = \sqrt{29}$

$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$

$= \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$

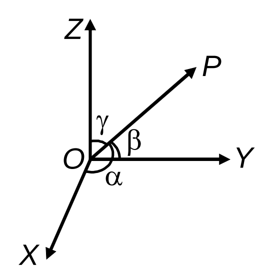
and $PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since *QR* = *PQ* + *PR*. Therefore the given points are collinear.

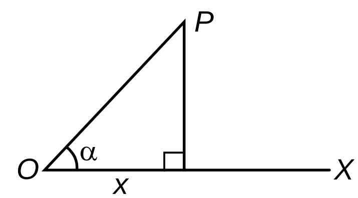
3. DIRECTION COSINES AND DIRECTION RATIOS

3.1 DIRECTION COSINES

If the position vector of a point *P* i.e., $\overrightarrow{OP}$ makes angles α, β and γ with the positive direction of *x*, *y* and *z* axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines. They are also denoted by *l*, *m* and *n* respectively.



i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.



It can be seen from the figure $\cos\alpha = \dfrac{x}{OP}$

Similarly, $\cos\beta = \dfrac{y}{OP}$ and $\cos\gamma = \dfrac{z}{OP}$

Where OP is the modulus of positive vector of P.

Clearly, $OP = \sqrt{x^2 + y^2 + z^2}$

so, $\quad l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{x^2 + y^2 + z^2}{OP^2} = 1$$

$\therefore \quad$ if $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\quad \hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

3.2 DIRECTION RATIOS

If a, b, c three numbers such that $\dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n}$

where l, m, n are direction cosines of a vector $\vec{r}$, then a, b, c are known as direction numbers or direction ratios of $\vec{r}$.

e.g., if $\hat{r} = 2\hat{i} - 3\hat{j} + 10\hat{k}$

then its direction ratios are 2, –3 and 10 or 4, –6 and 20 or any positive multiple of the components or direction cosines of $\vec{r}$.

Two vectors having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

They are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The angle between two vectors $\vec{r}_1 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r}_2 = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\cos\theta = \frac{\vec{r}_1 . \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}.\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Illustration 4

Question: **A vector $\vec{r}$ has length 21 and direction ratios 2, J3, 6. Find the vector $\vec{r}$.**

Solution: The direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{-3}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm\frac{6}{\sqrt{2^2 + (-3)^2 + 6^2}}$$

Since $\vec{r}$ makes an acute angle with x-axis, therefore $\cos\alpha > 0$ i.e., $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$\therefore \quad \vec{r} = 21\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \quad$ [using $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$]

or $\quad \vec{r} = 6\hat{i} - 9\hat{j} + 18\hat{k}$

So, components of $\vec{r}$ along *ox*, *oy* and *oz* are $6\hat{i}, -9\hat{j}$ and $18\hat{k}$ respectively.

***Illustration* 5**

Question: **Find the angle between the vectors with direction ratios 4, J3, 5 and 3, 4, 5.**

Solution: Let $\vec{a}$ = a vector parallel to the vector having direction ratios 4, –3, 5 = $4\hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{b}$ = a vector parallel to the vector having direction ratios 3, 4, 5 = $3\hat{i} + 4\hat{j} + 5\hat{k}$.

Let θ be the angle between the given vectors. Then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12 - 12 + 25}{\sqrt{16+9+25}\,\sqrt{9+16+25}} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}.$$

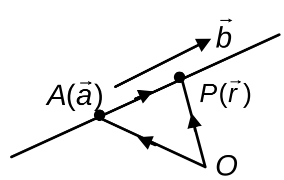
Thus, the angle between the vectors with direction ratios 4, –3, 5 and 3, 4, 5 is 60°.

4. EQUATION OF A STRAIGHTLINE IN SPACE

A straight line in space is specified basically in two ways viz., a line passing through a given point and parallel to a given vector and a line passing through two given points.

4.1 VECTOR EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Let *A* be a fixed point having position vector $\vec{a}$ and the line is parallel to the vector $\vec{b}$. *P* is an arbitrary point having position vector $\vec{r}$ on the line.



From ΔOAP, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b}$$

This is the required equation of line. λ is an arbitrary real number.

4.2 CARTESIAN EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND GIVEN DIRECTION RATIOS

Let A (a_1, a_2, a_3) be the fixed point and the line has direction ratios b_1, b_2, b_3.

Taking $\vec{r}$ as $x\hat{i} + y\hat{j} + z\hat{k}$ in the vector equation we see that

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} = \lambda$$

This is the Cartesian equation of the line also called symmetrical form of line. Any point on this line can be taken as

$$(a_1 + b_1\lambda,\ a_2 + b_2\lambda,\ a_3 + b_3\lambda)$$

b_1, b_2, b_3 can also be replaced by the direction cosines l, m, n of vector $\vec{b}$.

***Illustration* 6**

Question: **The Cartesian equations of a line are 6*x* J 2 = 3*y* + 1 = 2*z* J 2. Find its direction ratios and also find vector equation of the line.**

Solution: Recall that in the symmetrical form of line coefficients of x, y and z are unity. Therefore to put the given line in symmetric form, we must make the coefficients of x, y and z as unity.

The given line is 6*x* J 2 = 3*y* + 1 = 2*z* J 2

$$\Rightarrow \quad 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z-1)$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z-1}{3}$$

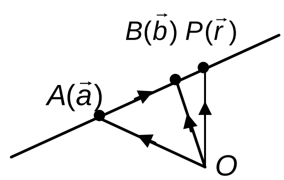
This shows that the given line passes through (1/3, –1/3, 1) and has direction ratios 1, 2, 3. In vector form this means that the line passes through the point having position vector $\vec{a} = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}$ and is parallel to the vector $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Therefore, its vector equation is

$$\vec{r} = \left(\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda\ (\hat{i} + 2\hat{j} + 3\hat{k}).$$

4.3 VECTOR EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Let A and B be two fixed points having position vectors $\vec{a}$ and $\vec{b}$. P is a variable point on the line.



From ΔOPA again, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\Rightarrow \quad \overrightarrow{OP} = \overrightarrow{OA} + \lambda(\overrightarrow{AB}) \Rightarrow \quad \vec{r} = \vec{a} + \lambda\,(\vec{b} - \vec{a})$$

This is the required equation.

4.4 CARTESIAN EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

If coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2), the Cartesian equation is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

***Illustration* 7**

Question: **Find the vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$, and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. Also, find the Cartesian equivalent of this equation.**

Solution: Let *A, B, C* be the points with position vectors $2\hat{i} + \hat{j} + \hat{k}, -\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

We have to find the equation of a line passing through the point A and parallel to $\overrightarrow{BC}$.

Now, $\overrightarrow{BC}$ = Position vector of C − Position vector of B

$$= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}$$

We know that the equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda\vec{b}.$$

Here, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$. So, the equation of the required line

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \qquad \text{...(i)}$$

Reduction to Cartesian form putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$$

$\Rightarrow$ $x = 2 + 2\lambda$, $y = -1 - 2\lambda$, $z = 1 + \lambda$

$\Rightarrow$ $\frac{x - 2}{2} = \frac{y + 1}{-2} = \frac{z - 1}{1}$, which is the Cartesian equivalent of (i).

4.5 ANGLE BETWEEN TWO LINES

If two lines are parallel to vectors $\vec{b}_1$ and $\vec{b}_2$, the angle between them is given by

$$\cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|}$$

Illustration 8

Question: **Find the angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$.**

Solution: The given equations are not in the standard form. The equations of the given lines can be written as

$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-2}{0}$$

and, $\frac{x-1}{1}=\frac{y+3/2}{3/2}=\frac{z+5}{2}$

Let $\vec{b}_1$ and $\vec{b}_2$ be vectors parallel to (i) and (ii) respectively,

then,

$$\vec{b}_1 = 3\hat{i}-2\hat{j}+0\hat{k} \text{ and } \vec{b}_2 = \hat{i}+\frac{3}{2}\hat{j}+2\hat{k}.$$

It θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(3)(1)+(-2)(3/2)+(0)(2)}{\sqrt{3^2+(-2)^2}\sqrt{1^2+(3/2)^2+2^2}} = 0$$

$\Rightarrow$ $\theta = \pi/2$.

5. INTERSECTION OF TWO LINES

Two lines in space can have the following three positions:

(i) They are parallel

(ii) They are intersecting

(iii) They are neither intersecting nor parallel. Such lines are called skew lines.

The following illustration shows how to check and find the point of intersection of two lines.

Illustration 9

Question: **Show that the lines $\vec{r}$ Z $(\hat{i}$ H $\hat{j}$ J $\hat{k})$ H a$(3\hat{i}$ J $\hat{j})$ and $\vec{r}$ Z $(4\hat{i}$ J $\hat{k})$ H «$(2\hat{i}$ H $3\hat{k})$ intersect. Find the point of intersection.**

Solution: The position vectors of arbitrary points on the given lines are $(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ $=(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$ and $(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$ respectively.

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$(3\lambda+1)\ i+(1-\lambda)\hat{j}-\hat{k}=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$$

$\Rightarrow$ $3\lambda + 1 = 2\mu + 4$, $1 - \lambda = 0$ and $-1 = 3\mu - 1$

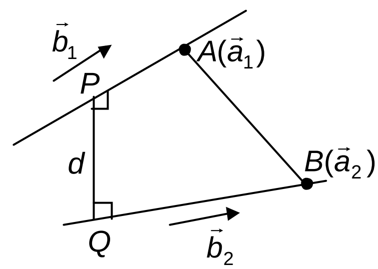
Solving last two of these three equations, we get $\lambda = 1$ and $\mu = 0$. These values of λ and μ satisfy the first equation. So, the given lines intersect. Putting $\lambda = 1$ in first line, we get

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} - \hat{j}) = 4\hat{i} + 0\hat{j} - \hat{k}$ as the position vector of the point of intersection. Thus, the coordinates of the point of intersection are (4, 0, –1).

6. SHORTEST DISTANCE BETWEEN TWO LINES

If $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ are two skew lines, the shortest distance between them is the perpendicular distance.

It is obtained as



$d = PQ$ = projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$

$= \overrightarrow{AB}.\hat{e}$

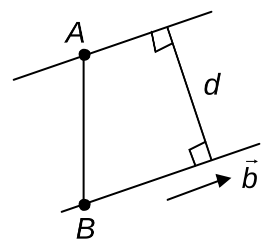
$$= \pm\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

More appropriately,

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{[\vec{b}_1\,\vec{b}_2\,(\vec{a}_2 - \vec{a}_1)]}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Clearly two lines intersect if $[\vec{b}_1\vec{b}_2\ (\vec{a}_2 - \vec{a}_1)] = 0$

If the lines are parallel,



i.e., $\vec{r} = \vec{a}_1 + \lambda\vec{b}$

and $\vec{r} = \vec{a}_2 + \lambda\vec{b}$

the formula to calculate shortest distance becomes

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

Illustration 10

Question: **Find the shortest distance between the lines**

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$.

Solution: We know that the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) . (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Comparing the given equations with the equations $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively, we have

$$\vec{a}_1 = 4\hat{i} - \hat{j},\ \ \vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}, \vec{b}_1 = i + 2j - 3\hat{k}$$

and $\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Now $\vec{a}_2 - \vec{a}_1 = -3\hat{i} + 0\hat{j} + 2\hat{k}$

and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0\hat{k}$

$\therefore$ $(\vec{a}_2 - \vec{a}_1) . (\vec{b}_1 \times \vec{b}_2) = (-3\hat{i} + 0\hat{j} + 2\hat{k}) . (2\hat{i} - \hat{j} + 0\hat{k})$

$= -6 + 0 + 0 = -6$

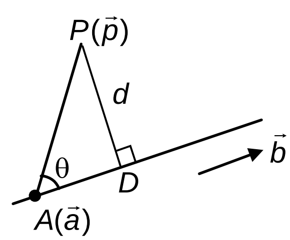
and $|\vec{b}_1 \times \vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ $d = \left| \frac{(\vec{a}_2 - \vec{a}_1) . (\vec{b}_2 \times \vec{b}_1)}{|\vec{b}_2 \times \vec{b}_1|} \right| = \left| \frac{-6}{\sqrt{5}} \right| = \frac{6}{\sqrt{5}}$

7. PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The perpendicular distance can be obtained using vector form as well as Cartesian form of the line.

Let the line be $\vec{r} = \vec{a} + \lambda\vec{b}$ and $P(\vec{p})$ be the point whose perpendicular distance is to be obtained.



$d = PD = AP \sin\theta$

$= \left| \frac{\overrightarrow{AP} \times \vec{b}}{|\vec{b}|} \right|$

$\Rightarrow \quad d = \left| \frac{(\vec{p} - \vec{a}) \times \vec{b}}{|\vec{b}|} \right|$

The following illustration shows how to obtain this distance using Cartesian form.

***Illustration* 11**

Question: **Find the foot of the perpendicular drawn from the point $2\hat{i} - \hat{j} + 5\hat{k}$ to the line $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$. Also find the length of the perpendicular.**

Solution: Let L be the foot of the perpendicular drawn from $P(2\hat{i} - \hat{j} + 5\hat{k})$ on the line

$$\vec{r} = 11\hat{i} - 2j - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}).$$

Let the position vector of L be

$$11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}) = (11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}.$$ Then

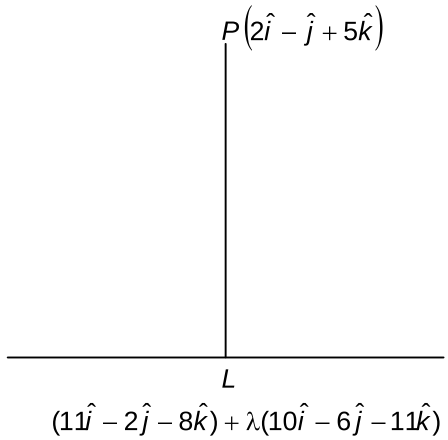
$\overrightarrow{PL}$ = position vector of L − position vector of P

$$= [(11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}] - [2\hat{i} - \hat{j} + 5\hat{k}]$$

$$= (9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k}$$

Since PL is perpendicular to the given line and the given line is parallel to $\vec{b} = 10\hat{i} - 4\hat{j} - 11\hat{k}$

$\therefore \quad \overrightarrow{PL} \perp \vec{b}$



$\Rightarrow \quad \overrightarrow{PL}.\vec{b} = 0 \Rightarrow [(9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k}].(10\hat{i} - 4\hat{j} - 11\hat{k}) = 0$

$\Rightarrow \quad 10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) = 0$

$\Rightarrow \quad 90 + 100\lambda + 4 + 16\lambda + 143 + 121\lambda = 0$

$\Rightarrow \quad 237\lambda = -237 \Rightarrow \lambda = -1$

Putting the value of λ, we obtain the position vector of L as $\hat{i} + 2\hat{j} + 3\hat{k}$

Now, $\overrightarrow{PL} = (i + 2\hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} + 5\hat{k}) = -\hat{i} + 3\hat{j} - 2\hat{k}$

Hence, length of the perpendicular from P on the given the $= |\overrightarrow{PL}| = \sqrt{1 + 9 + 4} = \sqrt{14}.$

8. IMAGE OF A POINT IN A LINE

Image means reflection of a point in a line behaving as a mirror. The steps are similar to that in previous article. First the foot of perpendicular is obtained then the image is obtained using mid-point formula.

***Illustration* 12**

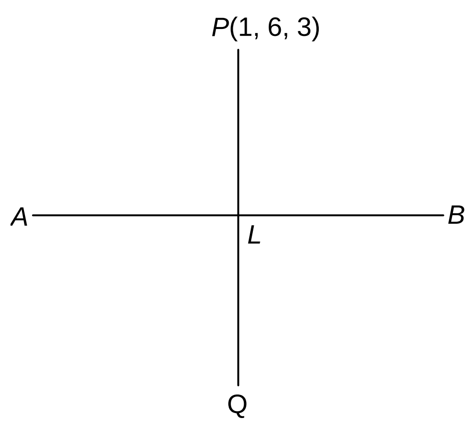
Question: **Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.**

Solution: Let P (1, 6, 3) be the given point, and let L be the foot of the perpendicular from P to the given line. The coordinates of a general point on the given line are given by

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$$

i.e., $x = \lambda$, $y = 2\lambda + 1$, $z = 3\lambda + 2$

Let the co-ordinates of L be $(\lambda, 2\lambda + 1, 3\lambda + 2)$



So, direction ratios of PL are $\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3$ *i.e.,* $\lambda - 1, 2\lambda - 5, 3\lambda - 1$. Direction ratios of the given line are 1, 2, 3 which is perpendicular to PL.

$\therefore$ $(\lambda - 1). 1 + (2\lambda - 5)2 + (3\lambda - 1) 3 = 0$

$\Rightarrow$ $14\lambda - 14 = 14 \Rightarrow \lambda = 1$.

so, coordinates of L are (1, 3, 5).

Let $Q(x_1, y_1, z_1)$ be the image of P(1, 6, 3) in the given line.

Then L is the mid-point of PQ.

$\therefore$ $\frac{x_1+1}{2} = 1, \frac{y_1+6}{2} = 3$ and $\frac{z_1+3}{2} = 5 \Rightarrow x_1 = 1, y_1 = 0$ and $z_1 = 7$.

Hence the image of P(1, 6, 3) in the given line is (1, 0, 7).

9. EQUATION OF PLANE IN VARIOUS FORMS

A plane can be defined as a surface on which if any two points are arbitrarily chosen, the line segment joining those points lies completely on the surface.

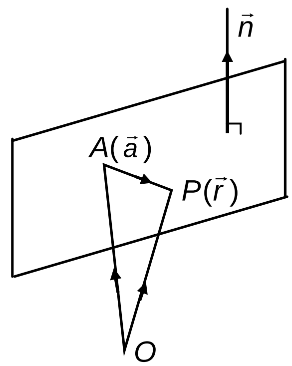
A plane is specified in various ways. We shall discuss them one by one below.

9.1 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND NORMAL TO A GIVEN VECTOR

Let a plane passes through $A(\vec{a})$ and is normal to a vector $\vec{n}$. If $P(\vec{r})$ is any arbitrary point on the plane, from figure we can write

$$\overrightarrow{AP} \cdot \vec{n} = 0$$

$$\Rightarrow \quad (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$



This is the vector equation of the plane also called scalar product form.

It can be converted to Cartesian from by putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and

$\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$.

We get $n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$

It can be seen that this a linear equation in x, y and z.

So, the Cartesian form of equation of plane is $ax + by + cz + d = 0$

where a, b, c denote the direction ratios of the normal vector.

Illustration 13

Question: **Find the equation in Cartesian form of the plane passing through the point (3, J3, 1) and normal to the line joining the points (3, 4, J1) and (2, J1, 5).**

Solution: We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to $\vec{n}$ is

$$(\vec{r} - \vec{a}).\vec{n} = 0 \quad \text{or} \quad \vec{r}.\vec{n} = \vec{a}.\vec{n}$$

Since the given plane passes through the point (3, –3, 1) and is normal to the line joining A (3, 4, –1) and B (2, –1, 5). Therefore

$$\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k} \quad \text{and} \quad \vec{n} = \overrightarrow{AB} = P.V. \text{ of } B - P.V. \text{ of } A$$

$$= (2\hat{i} - \hat{j} + 5\hat{k}) - (3\hat{i} + 4\hat{j} - \hat{k}) = -\hat{i} - 5\hat{j} + 6\hat{k}$$

Substituting $\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{n} = -\hat{i} - 5\hat{j} + 6\hat{k}$ in (i), we obtain

$$\vec{r}.(-\hat{i} - 5\hat{j} + 6\hat{k}) = (3\hat{i} - 3\hat{j} + \hat{k}).(-\hat{i} - 5\hat{j} + 6\hat{k})$$

or $\quad \vec{r}.(-\hat{i} - 5\hat{j} + 6\hat{k}) = -3 + 15 + 6$ or $\vec{r}.(-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$

This is the vector equation of the required plane. The Cartesian equation is

$$(x\hat{i} + y\hat{j} + z\hat{k}).\ (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$$

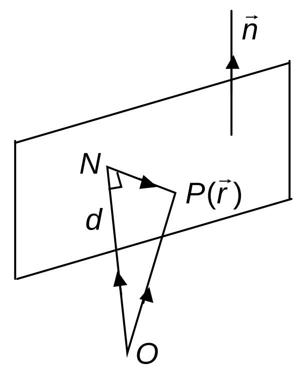
$\Rightarrow \quad -x - 5y + 6z = 18$ or $x + 5y - 6z + 18 = 0$

9.2 EQUATION OF A PLANE NORMAL TO A GIVEN VECTOR AND AT A GIVEN DISTANCE FROM ORIGIN

Let a plane be perpendicular to a unit vector $\hat{n}$ and at a perpendicular distance *d* from origin.

For an arbitrary point $P(\vec{r})$ on the plane, $\overrightarrow{NP}\,.\,\hat{n} = 0$

where *N* is the foot of perpendicular from *O* on the plane $(\overrightarrow{ON} = d\hat{n})$



$\therefore \quad (\vec{r} - d\hat{n})\,.\,\hat{n} = 0$

$\Rightarrow \quad \vec{r}\,.\,\hat{n} = d\hat{n}\,.\,\hat{n}$

$\Rightarrow \quad \vec{r}\,.\,\hat{n} = d$

This is called the normal form of equation of plane.

Obviously, this equation can also be converted to Cartesian form in the same way and which again leads to a linear equation in *x*, *y* and *z*.

Illustration 14

Question: **Reduce the equation $\vec{r}.(3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$ to normal form and hence find the length of perpendicular from origin to the plane.**

Solution: The given equation is $\vec{r}.(3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$

or $\quad \vec{r}.\vec{n} = 5$, where $\vec{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}$

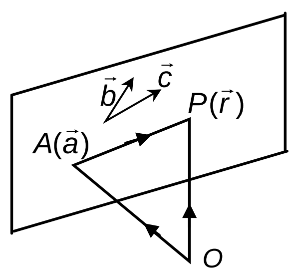
Since $|\vec{n}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13 \neq 1$, therefore the given equation is not in normal form. To reduce it to normal form, we divide both sides by $|\vec{n}|$ i.e.,

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|} \quad \text{or} \quad \vec{r} \cdot \left(\frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k}\right) = \frac{5}{13}$$

This is the normal form of the equation of given plane. The length of the perpendicular from the origin is $\frac{5}{13}$.

9.3 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN VECTORS

Let a plane pass through A $(\vec{a})$ and is parallel to the plane formed by two vectors $\vec{b}$ and $\vec{c}$. Since $\overrightarrow{AP}$ lies in the plane and $\vec{b}$ and $\vec{c}$ are two non-collinear vectors,



$$\overrightarrow{AP} = A\vec{b} + \mu\vec{c}$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b} + \mu\vec{c}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

Here λ and μ are arbitrary scalars.

This form is also called the parametric form of the plane. It can also be written in the non-parametric form as

$$(\vec{r} - \vec{a}) \cdot \vec{b} \times \vec{c} = 0$$

or $$[\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

***Illustration* 15**

Question: **Find the vector equation of the following plane in scalar product form:**

$$\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k}).$$

Solution: We know that the equation $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$. Here

$$\vec{a} = \hat{i} - \hat{j},\ \vec{b} = \hat{i} + \hat{j} + \hat{k} \ \textit{and} \ \hat{i} - 2\hat{j} + 3\hat{k}$$

The given planes is perpendicular to the vector

$$\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 2\hat{j} - 3\hat{k}$$

So, the vector equation of the plane in scalar product form is

$$\vec{r}\,.\,\vec{n} = \vec{a}\,.\,\vec{n}$$

$$\Rightarrow \quad \vec{r}\,.\,(5\hat{i} - 2\hat{j} - 3\hat{k}) = (\hat{i} - \hat{j})\,.\,(5\hat{i} - 2\hat{j} - 3\hat{k})$$

$$\Rightarrow \quad \vec{r}\,.\,(5\hat{i} - 2\hat{j} - 3\hat{k}) = (5 + 2 + 0) \quad \text{or} \quad \vec{r}.(5\hat{i} - 2\hat{j} - 3\hat{k}) = 7$$

9.4 EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

If three non-collinear points are given, there is a unique plane passing through them. Let the points be A, B and C having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lie in the plane. So, as in the previous article the equation of plane becomes

$$\vec{r} = \vec{a} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

or $$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

$$\Rightarrow \quad \vec{r} = (1 - \lambda - \mu)\ \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

***Illustration* 16**

Question: **Find the vector equation of the plane passing through the points**

$\hat{i}$ H $\hat{j}$ J $2\hat{k}$, $2\hat{i}$ J $2\hat{j}$ H $\hat{k}$, $\hat{i}$ H $2\hat{j}$ H $\hat{k}$.

Solution: Let A, B, C be the points with position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ respectively.

Then $\vec{AB}$ = P.V. or B – P.V. of A

$$= (2\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + \hat{j} - 2\hat{k}) = \hat{i} - 2\hat{j} + 3\hat{k}$$

and $\overrightarrow{AC} = P.V.$ of $\vec{C} - P.V.$ of $\vec{A}$

$$= \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right) = 0\,\hat{i} + \hat{j} + 3\hat{k}$$

A vector normal to the plane containing points A, B and C is

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix} = \left(-9\hat{i} - 3\hat{j} + \hat{k}\right)$$

The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and is normal to the vector $-9\hat{i} - 3\hat{j} + \hat{k}$. So its vector equation is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad \text{or} \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \quad \Rightarrow \quad \vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14$$

Cartesian equation of the plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ can be obtained as follows:

Since equation of plane is a linear equation, thereby any plane passing through a particular point (say $A(x_1, y_1, z_1)$) may be taken as

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where a, b, c are constant.

Now, substituting the coordinates of B and C we can find the equation of plane.

***Illustration* 17**

Question: **Find the equation of the plane through the points *A*(2, 2, J1), *B*(3, 4, 2) and *C*(7, 0, 6).**

Solution: The general equation of a plane passing through (2, 2, –1) is

$$a(x - 2) + b(y - 2) + c(z + 1) = 0 \qquad \ldots\text{(i)}$$

It will pass through B (3, 4, 2) and C(7, 0, 6) if

$$a(3 - 2) + b(4 - 2) + c(2 + 1) = 0 \text{ or } a + 2b + 3c = 0 \qquad \ldots\text{(ii)}$$

and $a(7 - 2) + b(0 - 2) + c(6 + 1) = 0$ or $5a - 2b + 7c = 0$ …(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{14 + 6} = \frac{b}{15 - 7} = \frac{c}{-2 - 10} \quad \text{or} \quad \frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda \text{ (say)}$$

$\Rightarrow$ $a = 5\lambda$, $b = 2\lambda$ and $c = -3\lambda$.

Substituting the values of a, b and c in (i), we get $5\lambda(x - 2) + 2\lambda(y - 2) - 3\lambda(z + 1) = 0$

or $5(x - 2) + 2(y - 2) - 3(z + 1) = 0$

$\Rightarrow$ $5x + 2y - 3z = 17$, which is the required equation of the plane.

9.5 INTERCEPT FORM OF A PLANE

This is a special case of the previous article. The equation of a plane intercepting the coordinate axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

***Illustration* 18**

Question: **Write the equation of the plane whose intercepts on the coordinate axes are J4, 2 and 3.**

Solution: We know that the equation of a plane whose intercepts on the coordinate axes are *a, b* and *c* respectively, is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Here, a = J4, b = 2, and c = 3. So, the equation of the required plane is

$$\frac{x}{-4}+\frac{y}{2}+\frac{z}{3}=1 \text{ or } -3x + 6y + 4z = 12.$$

10. ANGLE BETWEEN TWO PLANES

The angle between two planes is defined as the angle between their normals. If $\vec{n}_1$ and $\vec{n}_2$ are the normals and θ is the angle then $\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$

Obviously, two planes are perpendicular if $\vec{n}_1 \,.\, \vec{n}_2 = 0$.

They are parallel if $\vec{n}_1 = \lambda\vec{n}_2$ where λ is a scalar.

***Illustration* 19**

Question: **Find the angle between the planes *x* + *y* + 2*z* = 9 and 2*x* J *y* + *z* = 15.**

Solution: We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_1 = 0$ is given by $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Therefore, angle between $x + y + 2z = 9$ and $2x - y + z = 15$ is given by

$$\cos\theta=\frac{(1)\,(2)+(1)\,(-1)+(2)\,(1)}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+(-1)^2+1^2}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$$

11. FAMILY OF PLANES

11.1 PLANE PARALLEL TO A GIVEN PLANE

Since parallel planes have the same normal vector, so equation of a plane parallel to $\vec{r}.\hat{n}=d_1$ is of the form $\vec{r}.\hat{n}=d_2$, where d_2 is determined by the given conditions.

In Cartesian form, if $ax + by + cz + d = 0$ be the given plane then the plane parallel to this plane is $ax + by + cz + k = 0$.

Illustration 20

Question: **Find the equation of the plane through the point (1, 4, J2) and parallel to the plane J2*x* + *y* J 3*z* = 7.**

Solution: Let the equation of a plane parallel to the plane $-2x + y - 3z = 7$ be

$$-2x + y - 3z + k = 0 \qquad \text{…(i)}$$

This passes through (1, 4, –2), therefore $(-2)(1) + 4 - 3(-2) + k = 0$

$\Rightarrow \quad -2 + 4 + 6 + k = 0 \Rightarrow k = -8.$

Putting $k = -8$ in (i), we obtain

$$-2x + y - 3z - 8 = 0 \quad \text{or} \quad -2x + y - 3z = 8$$

This is the equation of the required plane.

11.2 PLANE PASSING THROUGH INTERSECTION OF TWO PLANES

Two planes intersect in a line if they are not parallel. Any plane through the line of intersection of two planes can be written as

$$(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0 \qquad \text{(vector form)}$$

or $$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \qquad \text{(Cartesian form)}$$

where λ is a real number.

Illustration 21

Question: **Find the equation of the plane containing the line of intersection of the plane *x* + *y* + *z* J 6 = 0 and 2*x* + 3*y* + 4*y* + 5 = 0 and passing through the point (1, 1, 1).**

Solution: The equation of a plane through the line of intersection of the given plane is

$$(x + y + z \text{ J } 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \qquad \text{...(i)}$$

If (i) passes through (1, 1, 1), we have $-3 + 14\lambda = 0 \Rightarrow \lambda = 3/14$.

Putting $\lambda = \frac{3}{14}$ in (i), we obtain the equation of the required plane as

$$(x + y + z \text{ J } 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0 \Rightarrow 20x + 23y + 26z \text{ J } 69 = 0.$$

12. DISTANCE OF A POINT FROM A PLANE

The perpendicular distance of a point $P(\vec{p})$ from the plane $\vec{r}.\vec{n} = d$ is given by $\frac{|\vec{p}.\vec{n} - d|}{|\vec{n}|}$

In Cartesian form, the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is equal to $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Illustration 22

Question: **Find the distance of the point (2, 1, 0) from the plane $2x + y + 2z + 5 = 0$.**

Solution: We know that the distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

So, required distance $= \frac{|2 \times 2 + 1 + 2 \times 0 + 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{10}{3}$.

13. DISTANCE BETWEEN PARALLEL PLANES

The distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Note that you must make the coefficients of x, y and z same in the equations of plane before applying this formula.

Illustration 23

Question: **Find the distance between the parallel planes 2*x* J *y* + 2z + 3 = 0 and 4*x* J 2*y* + 4*z* + 5 = 0.**

Solution: Let $P(x_1, y_1, z_1)$ be any point on 2*x* J *y* + 2*z* + 3 = 0, then

$2x_1 - y_1 + 2z_1 + 3 = 0$ ...(i)

The length of the perpendicular from $P(x_1, y_1, z_1)$ to 4*x* J 2*y* + 4*z* + 5 = 0 is

$$\left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right| = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \text{ [using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$.

14. BISECTORS OF TWO PLANES

In general, there are two angles between two planes. One is acute and other obtuse. If the equations of planes are $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$

or $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ the planes bisecting these angles are given by

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \text{ or } \vec{r}.(\hat{n}_1 \pm \hat{n}_2) = \frac{d_1}{|\vec{n}_1|} \pm \frac{d_2}{|\vec{n}_2|}$$

Illustration 24

Question: **Find the equations of the bisector planes of the angles between the planes *x* + 2*y* + 2*z* = 19 and 4*x* J 3*y* + 12*z* + 3 = 0 and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.**

Solution: The two given planes are

$$\vec{r}.(\hat{i}+2\hat{j}+2\hat{k})=19 \qquad ...(i)$$

and $$\vec{r}.(4\hat{i}-3\hat{j}+12\hat{k})+3=0 \qquad ...(ii)$$

The equations of the planes bisecting the angles between (i) and (ii) are

$$\frac{\vec{r}.(\hat{i}+2\hat{j}+2\hat{k})-19}{\sqrt{1^2+2^2+2^2}}=\pm\frac{\vec{r}.(4\hat{i}-3\hat{j}+12\hat{k})+3}{\sqrt{4^2+(-3)^2+144}}$$

or $$\frac{\vec{r}.(\hat{i}+2\hat{j}+2\hat{k})-19}{3}=\pm\frac{(\vec{r}.(4\hat{i}-3\hat{j}+12\hat{k})+3)}{13}$$

or $$\vec{r}.(13\hat{i}+26\hat{j}+26\hat{k})-247=\pm\,[\vec{r}.(12\hat{i}-9\hat{j}+36\hat{k})+9]$$

Taking positive sign on the RHS, we get $\vec{r}.(13\hat{i}+26\hat{j}+26\hat{k})-247=\vec{r}.(12\hat{i}=-9\hat{j}+36\hat{k})+9$

or $$\vec{r}.(\hat{i}+35\hat{j}-10\hat{k})-256=0 \qquad ...(iii)$$

and taking negative sign on the right hand side, we obtain

$$\vec{r}(25\hat{i}+17\hat{j}+62\hat{k})-238=0 \qquad ...(iv)$$

Hence, the two bisector planes are $\vec{r}.(\hat{i}+35\hat{j}-10\hat{k})=256$ and $\vec{r}.\,(25\hat{i}+17\hat{j}+62\hat{k})=238$

Now to obtain the angle bisector bisecting the acute angle between (i) and (ii), we find the angle between one of the given planes and one of the angle bisectors. Let θ be the angle between (i) and (iii) then

$$\cos\theta=\frac{(\hat{i}+2\hat{j}+2\hat{k}).(\hat{i}+35\hat{j}-10\hat{k})}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+(35)^2+(-10)^2}}=\sqrt{\frac{17}{78}}.$$

$$\therefore \quad \sin\theta=\sqrt{1-\frac{17}{78}}=\sqrt{\frac{61}{78}}.\ \text{So, }\tan\theta=\sqrt{\frac{61}{17}}>1$$

Thus, (iii) bisects the obtuse angle between (i) and (ii) and hence (iv) bisects the acute angle between the given planes.

15. LINE AND PLANE

15.1 UNSYMMETRICAL FORM OF LINE

The equation of two non-parallel planes taken together is called the unsymmetrical form of line because if a point lies in both these planes it has to lie on the line of intersection of the planes. The following illustration shows how to convert the unsymmetrical form of line to symmetrical form.

Illustration 25

Question: **Reduce in symmetrical form, the equation of the line of intersection two planes *x* J *y* + 2*z* = 5, 3*x* + *y* + *z* = 6.**

Solution: Let *a, b, c* be the direction ratios of the required line.

Since the required line lies in both the given planes, we must have

a J *b* + 2*c* = 0 and 3*a* + *b* + *c* = 0

Solving these two equations by cross-multiplication, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3} \text{ or } \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

In order to find a point on the required line, we put *z* = 0 in the two given equations to obtain

x J *y* = 5, 3*x* + *y* = 6

Solving these two equations, we obtain $x = \frac{11}{4}, y = \frac{-9}{4}$.

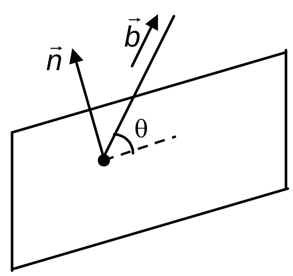
Therefore, coordinates of a point on the required line are (11/4, –9/4, 0). Hence, the equation of the required line is

$$\frac{x - \frac{11}{4}}{-3} = \frac{y - \left(-\frac{9}{4}\right)}{5} = \frac{z-0}{4} \text{ or } \frac{4x-11}{-12} = \frac{4y+9}{20} = \frac{z-0}{4}$$

or $$\frac{4x-11}{-3} = \frac{4y+9}{5} = \frac{z-0}{1}$$

15.2 ANGLE BETWEEN A LINE AND A PLANE

Let the line be $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane be $\vec{r}.\vec{n} = d$. If θ is the angle between them then



$$\cos(90° - \theta) = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

In Cartesian form, if the plane is *ax* + *by* + *cz* + *d* = 0 and line is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

then $$\sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

so, the condition that line is parallel to the plane is $\vec{b}.\vec{n} = 0$ or *al* + *bm* + *cn* = 0

and the condition of perpendicularity is $\vec{b} = \lambda\vec{n}$ or $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

Illustration 26

Question: **Find the angle between the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$ and the plane 2*x* + *y* J 3*z* + 4 = 0.**

Solution: The given line is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$ and the given plane is normal to the vector $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$. Therefore, the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|} = \frac{(3\hat{i} + 2\hat{j} + 4\hat{k}).(2\hat{i} + \hat{j} - 3\hat{k})}{\sqrt{3^2 + 2^2 + 4^2}\sqrt{2^2 + (1)^2 + (-3)^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{6+2-12}{\sqrt{29}\sqrt{14}} = \frac{-4}{\sqrt{406}} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right).$$

15.3 INTERSECTION OF LINE AND PLANE

If the line is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane is $ax + by + cz + d = 0$

the method is to take a general point on the line as $(x_1 + \lambda l, y_1 + \lambda m, z_1 + \lambda n)$ and put in the equation of plane to get value of λ and hence the point of intersection.

Illustration 27

Question: **Find the distance between the point with position vector $-\hat{i} - 5\hat{j} - 10\hat{k}$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ with the plane *x* J *y* + *z* = 5.**

Solution: The coordinates of any point on the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = r$ (say) are

(3*r* + 2, 4*r* J 1, 12*r* + 2)

If it lies on the plane *x* J *y* + *z* = 5, then $3r + 2 - 4r + 1 + 12r + 2 = 5 \Rightarrow 11r = 0 \quad \Rightarrow r = 0$.
Putting $r = 0$ in (i), we obtain (2, –1, 2) as the coordinates of the point of intersection of the given line and plane.
Required distance = distance between points (–1, –5, –10) and (2, –1, 2)

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = 13.$$

15.4 CONDITION FOR A LINE TO LIE IN A PLANE

Vector form : $\vec{a}\,.\,\vec{n} = d$ and $\vec{b}\,.\,\vec{n} = 0$

Cartesian form : $ax_1 + by_1 + cz_1 + d = 0$ and $al + bm + cn = 0$

Illustration 28

Question: **Find the equation of the plane passing through the point (0, 7, −7) and containing the line** $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

Solution: Let the equation of a plane passing through (0, 7, –7) be

$$a(x-0)+b(y-7)+c(z+7)=0 \qquad \ldots(i)$$

The line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ passes through the point (–1, 3, –2) and has direction ratios –3, 2, 1. If (i) contains this line, it must pass through (–1, 3, –2) and must be parallel to the line. Therefore

$$a(-1)+b(3-7)+c(-2+7)=0$$

i.e., $a(-1)+b(-4)+c(5)=0 \qquad \ldots(ii)$

and $-3a+2b+1c=0 \qquad \ldots(iii)$

On solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{-14}=\frac{b}{-14}=\frac{c}{-14}$$

$\Rightarrow \quad \frac{a}{1}=\frac{b}{1}=\frac{c}{1}=\lambda$ (say)

$\Rightarrow \quad a=\lambda,\ b=\lambda,\ c=\lambda.$

Putting the values of *a*, *b*, *c* in (i), we obtain

$$\lambda(x-0)+\lambda(y-7)+\lambda(z+7)=0$$

$\Rightarrow \quad x+y+z=0$

This is the equation of the required plane.

15.5 COPLANARITY OF TWO LINES

If the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ are coplanar, then

$$[\vec{a}_1\ \vec{b}_1\ \vec{b}_2]=[\vec{a}_2\ \vec{b}_1\ \vec{b}_2]$$

and the equation of plane containing them is

$$[\vec{r}\ \vec{b}_1\ \vec{b}_2]=[\vec{a}_1\ \vec{b}_1\ \vec{b}_2] \quad \text{or} \quad [\vec{r}\ \vec{b}_1\ \vec{b}_2]=[\vec{a}_2\ \vec{b}_1\ \vec{b}_2]$$

In Cartesian form, if the lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

then condition of coplanarity is $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

and the equation of plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

***Illustration* 29**

Question: **Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ are coplanar. Also, find the plane containing these two lines.**

Solution: We know that the line $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing these two lines is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Here $x_1 = -1$, $y_1 = -3$, $z_1 = -5$, $x_2 = 2$, $y_2 = 4$, $z_2 = 6$, $l_1 = 3$, $m_1 = 5$, $n_1 = 7$, $l_2 = 1$, $m_2 = 4$, $n_2 = 7$.

$$\therefore \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

so, the given lines are coplanar.

The equation of the plane containing the line is $\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$

or $(x + 1)(35 - 28) - (y + 3)(21 - 7) + (z + 5)(12 - 5) = 0$ or x J $2y + z = 0$.

16. IMAGE OF A POINT IN A PLANE

The reflection or image of a point in a plane can be obtained by first obtaining the foot of perpendicular from that point on the plane and then applying mid-point formula.

The following illustration shows the procedure.

Illustration 30

Question: **Find the image of the point (3, J2, 1) in the plane 3*x* J *y* + 4*z* = 2.**

Solution: Let *Q* be the image of the point *P*(3, –2, 1) in the plane 3*x* J *y* + 4*z* = 2. Then *PQ* is normal to the plane. Therefore direction ratios of *PQ* are 3, –1, 4. Since *PQ* passes through *P* (3, –2, 1) and has direction ratios 3, –1, 4. Therefore equation of *PQ* is

$$\frac{x-3}{3}=\frac{y+2}{-1}=\frac{z-1}{4}=r \text{ (say)}$$

Let the coordinates of *Q* be (3*r* + 3, –*r* J 2, 4*r* + 1). Let *R* be the mid-point of *PQ.* Then *R* lies on the plane 3*x* J *y* + 4*z* = 2.

The coordinates of *R* are

$$\left(\frac{3r+3+3}{2},\frac{-r-2-2}{2},\frac{4r+1+1}{2}\right)$$

or $$\left(\frac{3r+6}{2},\frac{-r-4}{2},2r+1\right)$$

Since *R* lies on 3*x* J *y* + 4*z* = 2, therefore,

$$3\left(\frac{3r+6}{2}\right)-\left(\frac{-r-4}{2}\right)+4(2r+1)=2$$

⇒ 13*r* = –13 ⇒ *r* = J 1

∴ The image is (0, –1, –3).

PRACTICE PROBLEMS

PP1. If a line makes angles α, β, γ with *OX, OY, OZ* respectively, find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.

PP2. Find the direction cosines of a line segment joining the points $R(3, -4, -2)$ and $S(1, -2, -1)$.

PP3. Find the direction cosines of *x, y* and *z*-axis.

PP4. Find the values of '*a*' for which points $(8, -7, a)$, $(5, 2, 4)$ and $(6, -1, 2)$ are collinear.

PP5. Can $-\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}$ be the direction cosines of any directed line? Justify your answer.

PP6. Find the direction cosines of the bisector of the angle between the vectors $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$.

PP7. Find the equation of the plane which is parallel to *x*-axis and cuts intercepts 2 and 5 on *y* and *z*-axes respectively.

PP8. Find the vector equation of the plane whose cartesian form of the equation is $2x + 3y - 9z = 4$.

PP9. Find the angle between the planes $2x - 3y + 4z = 1$ and $-x + y = 4$.

PP10. Show that planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ are mutually perpendicular.

PP11. Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines.

PP12. Find the coordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 3 units from point $(1, -2, 3)$.

PP13. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{-1/2} = \frac{z-6}{-5}$ are perpendicular find the value of *k*.

PP14. Prove that if a plane has the intercepts 1, 2, 3 and is at a distance of $\frac{6}{7}$ units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

PP15. Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find the equation of the plane through the points *P*(1, 1, 0), *Q*(1, 2, 1) and *R*(–2, 2, –1).

Solution:

The general equation of a plane passing through *P*(1, 1, 0) is

$a(x-1)+b(y-1)+c(z-0)=0$ …(i)

It will pass through *Q*(1, 2, 1) and *R*(–2, 2, –1), if

$a.0+b.1+c.1=0$ …(ii)

and $a(-3)+b.1+c(-1)=0$ …(iii)

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{(1)(-1)-(1)(1)}=\frac{b}{(1)(-3)-0(-1)}=\frac{c}{(0)(1)-(1)(-3)}$$

$\Rightarrow \quad \frac{a}{-2}=\frac{b}{-3}=\frac{c}{3}=\lambda$ (say)

$\Rightarrow \quad a=-2\lambda,\ b=-3\lambda,\ c=3\lambda$

Substituting the values of *a, b* and *c* in (i), we get

$-2\lambda(x-1)-3\lambda(y-1)+3\lambda z=0 \quad \Rightarrow \quad -2(x-1)-3(y-1)+3z=0$

$\Rightarrow \quad 2x+3y-3z-5=0$

***Example* 2:**

A plane meets the coordinate axes in *A, B, C* such that the centroid of triangle *ABC* is the point (*p, q, r*). Show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$.

Solution:

Let the equation of the required plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ …(i)

Then the coordinates of *A, B* and *C* are $A(a,\ 0,\ 0)$, $B(0,\ b,\ 0)$ and $C(0,\ 0,\ c)$ respectively.

So, the centroid of triangle *ABC* is $\left(\frac{a}{3},\ \frac{b}{3},\ \frac{c}{3}\right)$.

But the coordinates of the centroid are $(p,\ q,\ r)$

$\therefore \quad p=\frac{a}{3},\ q=\frac{b}{3},\ r=\frac{c}{3} \quad \Rightarrow \quad a=3p,\ b=3q,\ c=3r$

Substituting the values of *a, b* and *c* in (i), we get the required plane as

$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \quad \Rightarrow \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

***Example* 3:**

Find the equation of the plane passing through the point $(1,\ -1,\ 2)$ having 2, 3, 2 as direction ratios of normal to the plane.

Solution:

Here the plane passes through the point having position vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and is normal to the vector $\vec{n} = 2\hat{i} + 3\hat{j} + 2\hat{k}$.

So, the vector equation of the plane is $(\vec{r} - \vec{a}).\vec{n} = 0$

$\Rightarrow\ \vec{r}.\vec{n} = \vec{a}.\vec{n} \quad \Rightarrow\ \vec{r}.(2\hat{i} + 3\hat{j} + 2\hat{k}) = (\hat{i} - \hat{j} + 2\hat{k}).(2\hat{i} + 3\hat{j} + 2\hat{k})$

$\Rightarrow\ \vec{r}.(2\hat{i} + 3\hat{j} + 2\hat{k}) = 2 - 3 + 4 \quad \Rightarrow\ \vec{r}.(2\hat{i} + 3\hat{j} + 2\hat{k}) = 3$

The Cartesian equation of the plane is

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} + 2\hat{k}) = 3$ [Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$]

$\Rightarrow\ 2x + 3y + 2z = 3$

Example 4:

Find the vector equation of the plane passing through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also, find the Cartesian equation of the plane.

Solution:

The required plane passes through the point $A(2, 2, -1)$ whose position vector is $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and is normal to the vector $\vec{n}$ given by

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

We have,

$\overrightarrow{AB} = (3\hat{i} + 4\hat{j} + 2\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\overrightarrow{AC} = (7\hat{i} + 0\hat{j} + 6\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = 5\hat{i} - 2\hat{j} + 7\hat{k}$

$\therefore\ \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = (14 + 6)\hat{i} - (7 - 15)\hat{j} + (-2 - 10)\hat{k} = 20\hat{i} + 8\hat{j} - 12\hat{k}$$

The vector equation of the required plane is $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\Rightarrow\ \vec{r}.(20\hat{i} + 8\hat{j} - 12\hat{k}) = (2\hat{i} + 2\hat{j} - \hat{k}).(20\hat{i} + 8\hat{j} - 12\hat{k})$

$\Rightarrow\ \vec{r}.(20\hat{i} + 8\hat{j} - 12\hat{k}) = 40 + 16 + 12$

$\Rightarrow\ \vec{r}.(20\hat{i} + 8\hat{j} - 12\hat{k}) = 68$

$\Rightarrow\ \vec{r}.(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

The Cartesian equation of the plane is given by $(x\hat{i} + y\hat{j} + 3\hat{k}).(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

$\Rightarrow\ 5x + 2y - 3z = 17$

Example 5:

Find the equation of a plane through the intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 3$ passing through the point $(2, 1, -2)$.

Solution:

The equation of a plane through the intersection of $\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 3$ is

$[\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) - 5] + \lambda[\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) - 3] = 0$

$\Rightarrow\ \vec{r}\,.[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-5-3\lambda=0$ ...(i)

If plane in (i) passes through $(2, 1, -2)$, then the vector $2\hat{i}+\hat{j}-2\hat{k}$ should satisfy it.

$\therefore\ (2\hat{i}+\hat{j}-2\hat{k}).[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-(5+3\lambda)=0$

$\Rightarrow\ 2(1+2\lambda)+1(3-\lambda)-2(-1+\lambda)-(5+3\lambda)=0$

$\Rightarrow\ -2\lambda+2=0\ \Rightarrow\ \lambda=1$

Putting $\lambda=1$ in (i), we get the required equation of the plane as $\vec{r}\,.(3\hat{i}+2\hat{j}+0\hat{k})=8$

***Example* 6:**

Show that the line whose vector equation is $\vec{r}=(2\hat{i}-2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r}\,.(\hat{i}+5\hat{j}+\hat{k})=5$. Also, find the distance between them.

Solution:

The given line passes through the point having position vector $\vec{a}=2\hat{i}-2\hat{j}+3\hat{k}$ and is parallel to the vector $\vec{b}=\hat{i}-\hat{j}+4\hat{k}$.

The given plane is normal to the vector $\vec{n}=\hat{i}+5\hat{j}+\hat{k}$

We have $\vec{b}\,.\vec{n}=(\hat{i}-\hat{j}+4\hat{k}).(\hat{i}+5\hat{j}+\hat{k})=1-5+4=0$

So, $\vec{b}$ perpendicular to $\vec{n}$

Hence, the given line is parallel to the given plane.

The distance between the line and the parallel plane is the distance between any point on the line and the given plane.

Since the line passes through the point $\vec{a}=2\hat{i}-2\hat{j}+3\hat{k}$.

Therefore distance between the lien and the plane

=length of perpendicular from $\vec{a}=2\hat{i}-2\hat{j}+3\hat{k}$ to the given plane

$$=\frac{\left|(2\hat{i}-2\hat{j}+3\hat{k}).(\hat{i}+5\hat{j}+\hat{k})-5\right|}{\sqrt{1^2+5^2+1^2}}=\frac{10}{\sqrt{27}}$$

***Example* 7:**

Find the equation of the straight line through the origin parallel to the line
$(b+c)x+(c+a)y+(a+b)z=k=(b-c)x+(c-a)y+(a-b)z$.

Solution:

Equation of straight line through origin is $\dfrac{x-0}{l}=\dfrac{y-0}{m}=\dfrac{z-0}{n}$

where $l(b+c)+m(c+a)+n(a+b)=0$

and $l(b-c)+m(c-a)+n(a-b)=0$

On solving, $\dfrac{l}{2(a^2-bc)}=\dfrac{m}{2(b^2-ca)}=\dfrac{n}{2(c^2-ab)}$

$\therefore$ equation of the straight line is $\dfrac{x}{a^2-bc}=\dfrac{y}{b^2-ca}=\dfrac{z}{c^2-ab}$

Example 8:

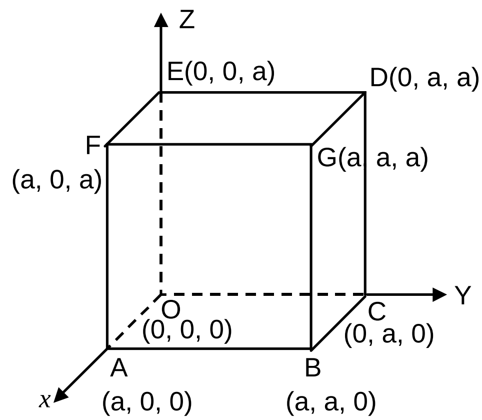
If a line makes angles ~, , ¥, ¢ with the diagonals of a cube, prove that

cos2~ + cos2 + cos2¥ + cos2¢ = J $\frac{4}{3}$.

Solution:

Let the length of each side of the cube be a.

The coordinates of the corners are as shown in the figure.



The four diagonals are *OG*, *AD*, *CF*, and *BE*.

The direction ratios of *OG* are

$a - 0, a - 0, a - 0$ or, a, a, a

Direction cosines of OG are

$$\frac{a}{\sqrt{3}\,a}, \frac{a}{\sqrt{3}\,a}, \frac{a}{\sqrt{3}\,a} \quad \text{or,} \quad \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Similarly, direction cosines of *AD* are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Direction cosines of *CF* are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

and direction cosines of *BE* are $\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with *OG*, *AD*, *CF*, *BE* respectively. Then,

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}}$$

$$\cos\beta = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{-l+m+n}{\sqrt{3}}$$

$$\cos\gamma = \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l-m+n}{\sqrt{3}}$$

$$\cos\delta = \frac{-l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{-l-m+n}{\sqrt{3}}$$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}\,[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m+n)^2]$$

$$= \frac{1}{3}[4\,(l^2 + m^2 + n^2) + 2lm + 2ln - 2lm - 2ln + 2mn - 2lm - 2mm + 2ln + 2lm - 2ln - 2mn]$$

$$= \frac{4}{3} \quad [\because l^2 + m^2 + n^2 = 1]$$

Now, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta$

$= 2\cos^2\alpha = 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 + 2\cos^2\delta - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos^2\delta) - 4$

$$= 2.\frac{4}{3} - 4 = -\frac{4}{3}$$

***Example* 9:**

Find the coordinates of those points on the line $\frac{x \text{ J } 1}{2} \text{ Z } \frac{y \text{ H } 2}{3} \text{ Z } \frac{z \text{ J } 3}{6}$ which is at a distance of 3 units from point (1, J2, 3).

Solution:

Given line is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ …(i)

Let $P \equiv (1, -2, 3)$

Direction ratios of line (i) are 2, 3, 6

$\therefore$ Direction cosines of line (i) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Equation of line (i) may be written as

$$\frac{x-1}{2/7} = \frac{y+2}{3/7} = \frac{z-3}{6/7} \qquad \text{…(ii)}$$

Coordinates of any point on line (ii) may be taken as

$$\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$$

Let $Q \equiv \left(\frac{2}{7}r - 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$

Distance of Q from $P = |\, r \,|$

According to question $|\, r \,| = 3 \quad \therefore \; r = \pm 3$

Putting the value of r, we have

$$Q \equiv \left(-\frac{1}{7}, -\frac{5}{7}, \frac{39}{7}\right) \text{ or, } Q \equiv \left(-\frac{13}{7}, -\frac{23}{7}, \frac{3}{7}\right)$$

***Example* 10:**

The direction cosine of the shortest distance between the lines $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z+1}{3}$ and $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-1}{2}$ are $\frac{-1}{\sqrt{42}},\frac{4}{\sqrt{42}},\frac{-5}{\sqrt{42}}$.

Find (i) its equation (ii) the points where it intersects the lines.

Solution:

Let $AB:\ \frac{x-2}{1}=\frac{y-3}{4}=\frac{z+1}{3}$ …(i)

and $CD:\ \frac{x-3}{2}=\frac{y-4}{3}=\frac{z-1}{2}$ …(ii)

Let PQ be the shortest distance between the lines AB and CD.

(i) Now, equation of the plane containing AB and PQ is

$$\begin{vmatrix} x-2 & y-3 & z+1 \\ 1 & 4 & 3 \\ -1 & 4 & -5 \end{vmatrix}=0 \quad \text{or, } -32(x-2)+2(y-3)+8(z+1)=0$$

or, $-32x+2y+8z+66=0$ or, $-16x+y+4z+33=0$

The equation of the plane containing CD and PQ is $\begin{vmatrix} x-3 & y-4 & z-1 \\ 2 & 3 & 2 \\ -1 & 4 & -5 \end{vmatrix}=0$

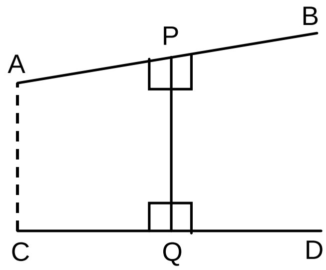
or, $-23(x-3)+8(y-4)+11(z-1)=0$ or $-23x+8y+11z+26=0$

The equation of the shortest distance is $-16x+y+4z+33=0$, $-23x+8y+11z+26=0$

(ii) Since P and Q lie on lines (i) and (ii)

Let $P \equiv (r+2, 4r+3, 3r-1)$

and $Q \equiv (2r'+3, 3r'+4, 2r'+1)$



The direction ratios of PQ are

$(x_1-x_2, y_1-y_2, z_1-z_2)$

i.e., $(r-2r'-1, 4r-3r'-1, 3r-2r'-2)$

But PQ is perpendicular to both AB and CD

$\therefore$ $1(r-2r'-1)+4(4r-3r'-1)+3(3r-2r'-2)=0$

and $2(r-2r'-1)+3(4r-3r'-1)+2(3r-2r'-2)=0$

or, $26r-20r'=11$ …(iii)

and $20r-17r'=9$

Solving (iii) and (iv), we get, $r=\frac{1}{6}$ and $r'=-\frac{1}{3}$

Hence $P\equiv\left(\frac{13}{6},\frac{11}{3},-\frac{1}{2}\right)$ and $Q\equiv\left(\frac{7}{3},3,\frac{1}{3}\right)$

EXERCISE – I

1. Reduce the equations of the following planes in intercept form and find its intercepts on the coordinates axes:
(i) $4x+3y-6z-12=0$ (ii) $2x+3y-z=6$
(iii) $2x-y+z=5$

2. Find the vector equation of a plane passing through a point having position vector $2\hat{i}-\hat{j}+\hat{k}$ and perpendicular to the vector $4\hat{i}+2\hat{j}-3\hat{k}$.

3. Find the vector and Cartesian equations of a plane passing through the point $(1, -1, 1)$ and normal to the line joining the points $(1, 2, 5)$ and $(-1, 3, 1)$.

4. Determine the value of λ for which the following planes are perpendicular to each other.
(i) $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})=7$ and $\vec{r}.(\lambda\hat{i}+2\hat{j}-7\hat{k})=26$
(ii) $2x-4y+3z=5$ and $x+2y+\lambda z=5$
(iii) $3x-6y-2z=7$ and $2x+y-\lambda z=5$

5. Find the equation of the plane passing through the line of intersection of the planes $2x-7y+4z-3=0$, $3x-5y+4z+11=0$ and the point $(-2, 1, 3)$.

6. Find the points on the lines $\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}$ and $\frac{x}{-3}=\frac{y-9}{2}=\frac{z-2}{4}$ which are nearest to each other.

7. A line with direction ratios (2, 7, –5) is drawn to intersect the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ and $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$. Find the coordinates of the points of intersection and the length intercepted on it.

8. Find the angel between the lines:
(a) $\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$
(b) $\vec{r}=4\vec{i}-\vec{j}+\lambda(\vec{i}+2j-2\vec{k})$ and $\vec{r}=\vec{i}-\vec{j}+2\vec{k}-\mu(2\vec{i}+4\vec{j}-4\vec{k})$

9. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=2$.

10. Find the equation of the plane which passes through the point $(3, 4, -1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also find the distance between the two planes.

EXERCISE – II

1. A plane meets the coordinate axes at A, B and C respectively such that the centroid of triangle ABC is $(1,\,-2,\,3)$. Find the equation of the plane.

2. A plane passes through the point $(1,\,-2,\,5)$ and is perpendicular to the line joining the origin to the point $3\hat{i}+\hat{j}-\hat{k}$. Find the vector and Cartesian forms of the equation of the plane.

3. Write the normal form of the equation of the plane $2x-3y+6z+14=0$.

4. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined with the coordinate axes.

5. Find the vector equation of the plane passing through the points $(1,\,1,\,1)$, $(1,\,-1,\,1)$ and $(-7,\,-3,\,-5)$.

6. Find the equation of a plane passing through the point $(-1,\,-1,\,2)$ and perpendicular to the planes $3x+2y-3z=1$ and $5x-4y+z=5$

7. Find the equation of the plane through the line of intersection of the planes $\vec{r}\,.\left(\hat{i}+3\hat{j}\right)+6=0$ and $\vec{r}\,.\left(3\hat{i}-\hat{j}-4\hat{k}\right)=0$, which is at a unit distance from the origin.

8. Find the equation of the plane passing through the intersection of the planes $2x+3y-z+1=0$ and $x+y-2z+3=0$ and perpendicular to the plane $3x-y-2z-4=0$.

9. Find the coordinates of the foot of perpendicular from the point $(2,\,6,\,3)$ to the line $\frac{x}{2}=\frac{y-1}{2}=\frac{z-2}{3}$. Also find the equation of this perpendicular.

10. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r}=3\vec{i}+8\vec{j}+3\vec{k}+\lambda\left(3\vec{i}-\vec{j}+\vec{k}\right)$ and $\vec{r}=-3\vec{i}-7\vec{j}+6\vec{k}+\mu\left(-3\vec{i}+2\vec{j}+4\vec{k}\right)$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. –1

PP2. $\left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

PP3. 1, 0, 0; 0, 1, 0 and 0, 1, 0

PP4. –2

PP6. $\frac{1}{3\sqrt{6}}, -\frac{7}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}$

PP7. $5y + 2z = 10$

PP8. $\vec{r} = \left(2\hat{i} + 3\hat{j} - 9\hat{k}\right) = 4$

PP9. $\theta = \cos^{-1}\left(\frac{5}{\sqrt{58}}\right)$ or $\pi - \cos^{-1}\left(\frac{5}{\sqrt{58}}\right)$

PP12. $\left(-\frac{1}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ or $\left(-\frac{13}{7}, \frac{-23}{7}, \frac{3}{7}\right)$

PP13. –1

ANSWERS TO EXERCISE – I

1. (i) $\frac{x}{3}+\frac{y}{4}=\frac{z}{-2}=1;\ 3,\ 4,\ -2$

(ii) $\frac{x}{3}+\frac{y}{2}+\frac{z}{-6}=1;\ 3,\ 2,\ -6$

(iii) $\frac{x}{5/2}+\frac{y}{-5}+\frac{z}{5}=1;\ \frac{5}{2},\ -5,\ 5$

2. $\vec{r}\,.\left(4\hat{i}+2\hat{j}-3\hat{k}\right)=3$

3. $\vec{r}\,.\left(2\hat{i}-\hat{j}+4\hat{k}\right)=7,\ \ 2x-y+4z=7$

4. (i) 17

(ii) 2

(iii) 0

5. $15x-47y+28z=7$

6. (3, 8, 3) and (–3, –7, 6)

7. $(2,\ 8,\ 3),\ (0,\ 1,\ 2);\ \sqrt{78}$

8. (a) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$

(b) 0°

9. $x+y+z=a+b+c$

10. $2x-3y+5z+11=0\ ,\ \frac{9}{\sqrt{78}}$

ANSWERS TO EXERCISE – II

1. $6x - 3y + 2z = 18$

2. $\vec{r}\,.\left(3\hat{i} + \hat{j} - \hat{k}\right) = -4,\ 3x + y - z = 2$

3. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$

4. $x + y + z = 9$

5. $\vec{r}\,.\left(3\hat{i} - 4\hat{j}\right) + 1 = 0$

6. $5x + 9y + 11z - 8 = 0$

7. $\vec{r}\,.\left(-2\hat{i} + 4\hat{j} + 4\hat{k}\right) + 6 = 0$ or $\vec{r}\,.\left(4\hat{i} + 2\hat{j} - 4\hat{k}\right) + 6 = 0$

8. $7x + 13y + 4z - 9 = 0$

9. $(2, 3, 5)$ and $\frac{x-2}{0} = \frac{y-3}{-3} = \frac{z-5}{2}$

10. $3\sqrt{30}$ units or $\vec{r} = 3\vec{i} + 8\vec{j} + 3\vec{k} + t\left(6\vec{i} + 15\vec{j} - 3\vec{k}\right)$