LESSON 8

AREAS BOUNDED BY CURVES

1. APPLICATION OF INTEGRATION TO AREAS

Definite integral is used to evaluate areas bounded by curves. To do problems under this heading, one must be able to draw a rough figure of the curve when the equation is given. Some rules about drawing curves are given below. Familiar curves like lines, circles and conics are not discussed here.

Guidelines

(i) Check whether the curve is symmetrical about the x-axis or not. The curve is symmetrical about the x-axis, if its equation is unchanged when y is replaced by $-y$.

(ii) The curve is symmetrical about the y-axis if its equation is unchanged when x is replaced by $-x$.

(iii) Put $y = 0$ in the equation of the curve. This will give the points where it cuts the x-axis

(iv) Put $x = 0$ in the equation of the curve. This will give the points where it cuts the y-axis.

(v) The curve is symmetrical about the line $y = x$ if its equation does not change when x and y are interchanged.

(vi) Find the turning points of the graph by equating $\frac{dy}{dx} = 0$

(vii) Find the intervals of curve in which it increases and decreases if required.

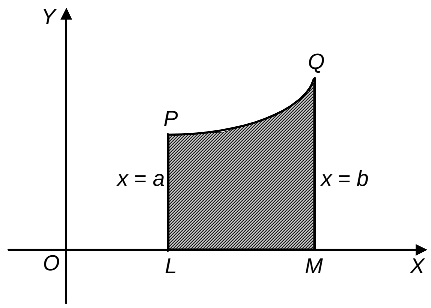
(viii) Use periodicity wherever possible.

(ix) Check behaviour at $x \to \pm\infty$ and $y \to \pm\infty$.

2. ESTIMATION OF AREAS

Four cases are discussed below:

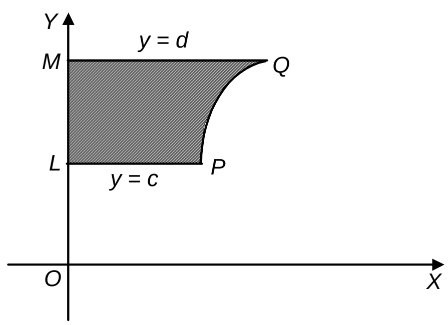
Case I : *PQ* is an arc of a curve whose equation is $y = f(x)$. We have an area bounded by *PQ* on one side; by the *x*-axis on another and the two parallel lines $x = a$ and $x = b$ (shown by *PL* and *QM*), $a < b$.



The area $PLMQ = \int_{x=a}^{x=b} y\,dx = \int_a^b f(x)\,dx$

Case II: *PQ* is an arc of a curve whose equation is $y = f(x)$ or $x = f(y)$.

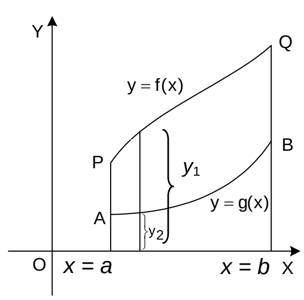
In this case y-axis is one boundary and the other two are the lines $y = c$ and $y = d$.



The area $LPQM = \int_{y=c}^{y=d} x\,dy = \int_c^d f(y)\,dy$

In this case the integration is with respect to *y*.

Case III: The figure encloses an area between two curves one of which is represented by *PQ* with equation $y = f(x)$ and the other by *AB* with the equation $y = g(x)$.

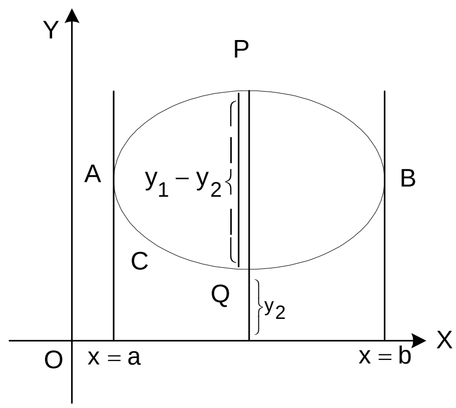


Area $PABQ = \int_a^b (y_1 - y_2)\,dx$ where $y_1 = f(x)$ and $y_2 = g(x)$

$$= \int_a^b \{f(x) - g(x)\}\,dx$$

Case IV: The figure represents the region bounded by a closed curve ACQBP.

The area of the region bounded by a closed curve ACQBP is $\int_a^b (y_1 - y_2)\,dx$, $y_1 > y_2$



The values of y_1 and y_2 are obtained by solving the equation of the curve as a quadratic in y whose larger root y_1 and smaller root y_2 are functions of x.

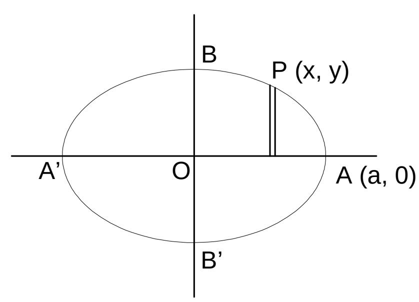
a and b are the coordinates of the points of contact of tangents drawn parallel to the y-axis.

***Illustration* 1**

Question: **Find the area of the ellipse** $\frac{x^2}{a^2}$ H $\frac{y^2}{b^2}$ J **1** Z **0**

***Solution*:** The ellipse is symmetrical about both axes and hence the area enclosed

= 4 (area of the quadrant)



$$= 4\int_0^a y\,dx$$

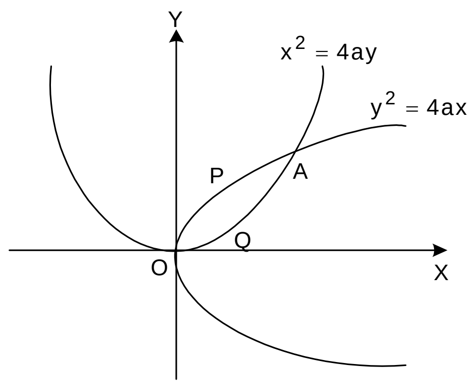
$$= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \frac{4b}{a}\left[\frac{a^2\pi}{4}\right] = \pi ab \text{ sq. units}$$

Illustration 2

Question: **Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.**

Solution: The two parabolas intersect at O(0,0) and *A* (4*a*, 4*a*). The area included between the two curves = area O*QAP*.



$$= \int_{x=0}^{x=4a} (y_1 - y_2)\, dx$$

$$= \int_0^{4a} \left(2\sqrt{a}\,\sqrt{x} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3} x^{3/2} - \frac{x^3}{12a} \right]_0^{4a} = 4\frac{\sqrt{a}}{3}\, 8.a^{3/2} - \frac{64a^3}{12a} = \frac{16a^2}{3} \text{ sq. units.}$$

Note: Sometimes it is better to use the formula $\int_c^d x\,dy$ instead of $\int_a^b y\,dx$ in the computation of area to simplify calculations, as the following illustration shows.

PRACTICE PROBLEMS

PP1. The area of the region enclosed by $y = 2 - x^2$ and $y = -x$ is ……………

PP2. Obtain area bounded by $x = y^2$ and x-axis between $x = 0$ and $x = 9$.

PP3. Sketch the curves $y = x^2 - 4$ and $y = -x^2 - 2x$ and find area of region bounded by them.

PP4. Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.

PP5. Sketch the region bounded by $y = 2x - x^2$ and x-axis and find its area using integration.

PP6. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

PP7. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

PP8. Using integration, find the area of the region bounded by the line $2y = -x + 8$, x-axis and the lines $x = 2$ and $x = 4$.

PP9. Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

PP10. Sketch the graph $y = | x + 1 |$. Evaluate $\int_{-3}^{1} |x + 1|\, dx$. What does this value represent on the graph?

SOLVED SUBJECTIVE EXAMPLES

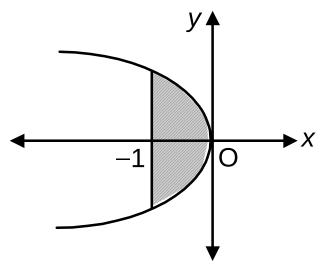
***Example* 1:**

Find the area bounded by $y^2 = -4x$ and its latus-rectum.

Solution:

The required area = $2\int_{-1}^{0}\sqrt{-4x}\,dx$

$= -\frac{8}{3}(-x)^{3/2}\Big|_{-1}^{0} = \frac{8}{3}$ sq. units.



***Example* 2:**

Find the area bounded by the curves $y = x$ and $y = x^3$.

Solution:

The equations of the given curves are

$y = x$ …(i)

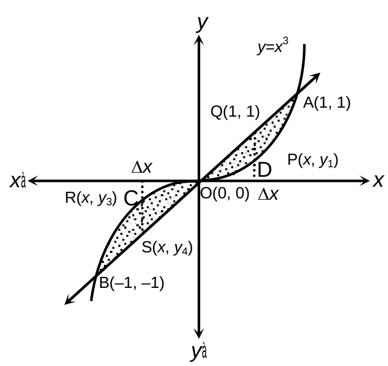
and $y = x^3$ …(ii)

Clearly, $y = x$ is a line passing through the origin and making an angle of 45° with x-axis.

The shaded portion shown in figure:

Solving $y = x$ and $y = x^3$ simultaneously, find that the two curves intersect at O, (0, 0), A(1, 1) and B(–1, –1).

$\therefore$ Required area = area $BCOB$ + area $ODAO$



Area $BCOB$:

So the approximating rectangle shown in this region has length $= (y_4 - y_3)$, width Δx and

area $= (y_4 - y_3)\Delta x$

Since the approximating rectangle can move from $x = -1$ to $x = 0$

$\therefore$ Area $BCOB = \left|\int_{-1}^{0}(y_4 - y_3)dx\right| \Rightarrow \left|\int_{-1}^{0}\left(x - x^3\right)dx\right|$

[$\because$ $R(x, y_3)$ and $S(x, y_4)$ lie on (ii) and (i) respectively $\therefore$ $y_3 = x^3$ and $y_4 = x$]

Area $ODAO$

So the approximating rectangle shown in this region has length $= (y_2 - y_1)$, width Δx and area $= (y_2 - y_1)\Delta x$

$\therefore$ Area $ODAO = \int_{0}^{1}(y_2 - y_1)dx = \int_{0}^{1}\left(x - x^3\right)$

[$\because$ $P(x, y_1)$ and $Q(x, y_2)$ lie on (ii) and (i) respectively, $\therefore$ $y_1 = x^3$ and $y_2 = x$]

Hence required area $= \left|\int_{-1}^{0}\left(x - x^3\right)dx\right| + \int_{0}^{1}\left(x - x^3\right)dx = \left|\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{0}\right| + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$

Required area $= \left|-\left(\frac{1}{2} - \frac{1}{4}\right)\right| + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$ sq. unit

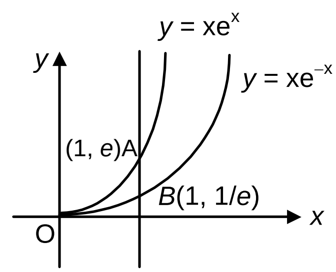
Example 3:

Find the area between $y = xe^x$ and $y = xe^{\text{J}x}$ and the line $x = 1$.

Solution:

The line $x = 1$ meets the curves in $A(1, e)$ and $B(1, 1/e)$. Both the curves pass through origin.

The required area



$$A = \int_{0}^{1}(y_1 - y_2)\, dx = \int_{0}^{1}(xe^x - xe^{-x})\, dx$$

$$= \left[x\left\{e^x + e^{-x}\right\}\right]_0^1 - \int_{0}^{1}(e^x + e^{-x})\, dx$$

$$= \left(e + \frac{1}{e}\right) - \left[e^x - e^{-x}\right]_0^1 = \left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right) = \frac{2}{e} \text{ sq. units.}$$

Example 4:

Find the area bounded by the curve, $y = f(x) = x^4$ J $2x^3 + x^2 + 3$, the x-axis and the ordinates corresponding to the minimum of function $f(x)$.

Solution:

We have $f(x) = x^4 - 2x^3 + x^2 + 3$

To find minimum $f'(x) = 4x^3 - 6x^2 + 2x = 0$

$\Rightarrow$ $2x(x - 1)(2x - 1) = 0$ $\Rightarrow$ $x = 0, 1, \frac{1}{2}$

$f''(x) = 12x^2 - 12x + 2 = 2(6x^2 - 6x + 1)$

$f''(x)_{x=0} > 0$ $\Rightarrow$ minimum exists at $x = 0$

$f''(x) < 0$ $\Rightarrow$ maximum exists at $x = \frac{1}{2}$

$f''(x)_{x=1} > 0$ $\Rightarrow$ minimum exists at $x = 1$

$\Rightarrow$ The curve is bounded by the ordinates $x = 0$ and $x = 1$

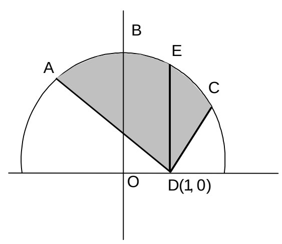
$\therefore$ Required area = $\int_0^1 (x^4 - 2x^3 + x^2 + 3)\, dx = \left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + 3x\right)\Bigg|_0^1 = \frac{91}{30}$ sq. units.

***Example* 5:**

Sketch the region bounded by the curves, $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area.

Solution:

$y = \sqrt{5 - x^2}$ represents the upper part of the circle $y^2 + x^2 = 5$



$y = |x - 1|$ consists of two equations $y = x - 1$ for $x > 1$ and $y = 1 - x$ for $x < 1$. Points of intersection $A = (-1, 2)$ and $C = (2, 1)$

Required area = Area ADEA + Area EDCE

$$= \int_{-1}^{+1} \left\{\sqrt{5 - x^2} - (1 - x)\right\} dx + \int_1^2 \left\{\sqrt{5 - x^2} - (x - 1)\right\} dx$$

Note: The ordinate of point on the segment AD is $1 - x$ and on the segment DC is $x - 1$.

$$= 2\int_0^1 \left(\sqrt{5 - x^2} - 1\right) dx + \int_1^2 \left(\sqrt{5 - x^2} - (x - 1)\right) dx$$ (**Note:** x is an odd function)

$$= 2\left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - x\right]_0^1 + \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - \frac{x^2}{2} + x\right]_1^2$$

$$= 2\left[\frac{2}{2} + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - 1\right] + \left[1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - 2 + 2\right] - \left[\frac{2}{2} + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\right]$$

$$= \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - \frac{1}{2}$$

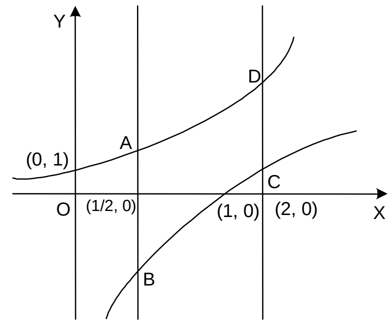
Example 6:

Sketch the curves and identify the region bounded by $x = \frac{1}{2}, x = 2, y = \log_e x$ and $y = 2^x$. Find the area of the region.

Solution:

(i) $y = \log_e x$

This curve passes through $(1, 0)$; for $x > 1$, $\log_e x > 0$ and the graph is above the X-axis. For $x < 1, \log_e x < 0$ and the corresponding graph is below the x-axis. As $x \to 0, \log_e x \to -\infty$. Therefore the curve is asymptotic with negative Y-axis.



It meets $x = \frac{1}{2}$ *at* $B\left(\frac{1}{2}, \log_e \frac{1}{2}\right)$ and $x = 2$ at $C\,(2, \log_e 2)$

(ii) $y = 2^x$

This curve passes through (0, 1). 2^x is always positive and therefore the graph is above the x-axis. As $x \to -\infty, 2^x \to 0.$ Therefore the negative X-axis is an asymptote for the curve.

It meets $x = \frac{1}{2}$ *at* $A\left(\frac{1}{2}, 2^{1/2}\right)$ and $x = 2$ at D (2, 4)

Required = area of region $ABCD$

$$= \int_{1/2}^{2} \left(2^x - \log_e x\right) dx \;= \left[\frac{2^x}{\ln 2} - (\,x \ln x - x)\right]_{1/2}^{2}$$

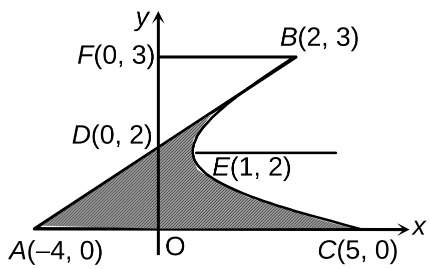
$$= \frac{4-\sqrt{2}}{\ln 2} - 2\ln 2 + 2 - \frac{1}{2} + \frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{4-\sqrt{2}}{\ln 2} - \frac{5}{2}\ln 2 + \frac{3}{2}$$

Example 7:

Find the area of the region bounded by the parabola $(y - 2)^2 = (x - 1)$, the tangent drawn to it at the point whose ordinate is 3 and the X-axis.

Solution:

$(y - 2)^2 = (x - 1)$ represents a parabola with vertex at $E\,(1, 2)$. Point on the parabola with ordinate 3 is $(2, 3)$



Differentiating

$$2(y-2)\frac{dy}{dx}=1$$

$\therefore$ Slope of the tangent at $B(2,3)$ is $\frac{1}{2}$

Equation of the tangent at B is

$$y-3=\frac{1}{2}(x-2)$$

i.e. $x-2y+4=0$

This meets the x-axis at $A(-4,0)$ and y-axis at $D(0,2)$

The parabola itself meets the x-axis at $C(5,0)$

Required area = shaded in the Figure

$= \text{Area of } \Delta AOD + \text{ Area of region OCEBDO}$

$= \frac{1}{2}\times 4\times 2 + \text{ Area of OCEBFO – Area of } \Delta \text{ DBF}$

$= 4-\frac{1}{2}\times 2\times 1+\int\limits_{y=0}^{3}\{(y-2)^2+1\}\,dy$ using the formula $\int x\,dy$

$= 3+\int\limits_{0}^{3}\left(y^2-4y+5\right)dy$

$= 3+\left[\frac{y^3}{3}-2y^2+5y\right]_0^3 = 3+(9-18+15)=3+6=9$

***Example* 8:**

Find the total area enclosed by the curve $a^2y^2 \; Z \; x^2(a^2 \; J \; x^2)$.

Solution:

The following points may be noted with reference to the curve.

(i) Changing x to $-x$ does not alter the equation. Hence the curve has symmetry about the y-axis.

(ii) Equally, it has symmetry about the x-axis also.

(iii) The curve intersects the x-axis at $x = -a$, $x = 0$, $x = a$.

(iv) $a^2y^2 > 0$ and hence $x^2(a^2 - x^2) > 0$

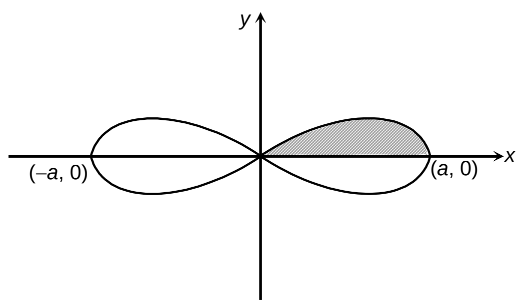
$\therefore$ $x^2 < a^2$ i.e. $-a < x < a$

$\therefore$ The curve gets bounded at ($-a$, 0) and (a, 0) beyond which on either side of the x-axis the curve does not exist.

(v) The only point where it intersects the y-axis is at the origin.

(vi) Hence the curve may be found to have two parts (each part is called a loop) on either side of the y-axis.

The curve is as shown below:



Area of both the loops = 4 $\times$ shaded area (by symmetry)

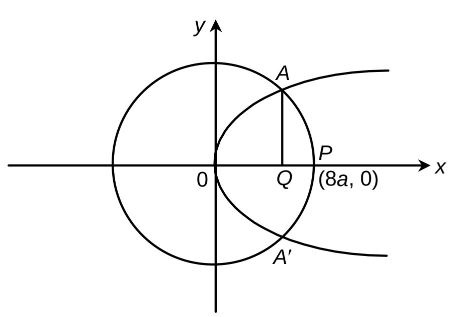
$$= 4\int_0^a y\,dx = \frac{4}{a}\int_0^a x\sqrt{a^2 - x^2}\,dx = -\frac{2}{a}\cdot\frac{2}{3}\left[\left(a^2 - x^2\right)^{3/2}\right]_0^a = \frac{4}{3a}\cdot a^3 = \frac{4a^2}{3}$$

***Example* 9:**

Show that the area enclosed by the circle $x^2 + y^2 = 64a^2$ and the parabola $y^2 = 12ax$ is $a^2\left(\frac{16}{\sqrt{3}} + \frac{64\pi}{\sqrt{3}}\right)$.

Solution:

Solving these equations we get the points of intersection as $A\ (4a, 4\sqrt{3}\ a)$ and $A\ (4a, -4\sqrt{3}a)$



Required area = *OAPAàO*

$$= 2\,[\text{Area } OPAO] \;= 2\,[\text{area } OQAO + \text{area } QPAQ]$$

$$= 2\left[\int_0^{4a} 2\sqrt{3ax}\,dx + \int_{4a}^{8a}\sqrt{64a^2 - x^2}\,dx\right]$$

$$= 4\sqrt{3}\,\sqrt{a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{4a} + 2\left\{\frac{x}{2}\sqrt{64a^2 - x^2} + \frac{64a^2}{2}\sin^{-1}\left(\frac{x}{8a}\right)\right\}_{4a}^{8a}$$

$$= \frac{8\cdot 8\sqrt{3a}}{3}\,a\sqrt{a} + 2\left[-2\cdot 4\sqrt{3}a^2 + 32a^2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{64\,a^2\sqrt{3}}{3} - 16\,a^2\sqrt{3} + 64a^2\,\frac{2\pi}{6} = \frac{16a^2}{\sqrt{3}} + \frac{64a^2\pi}{3} = a^2\left(\frac{16}{\sqrt{3}} + \frac{64\pi}{3}\right) \text{ sq.units}$$

***Example* 10:**

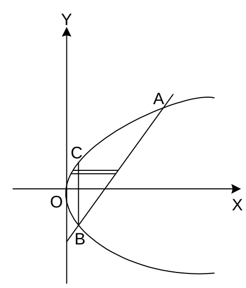
Find the area of the segment cut off from the parabola $y^2 = 2x$ by the line $y = 4x - 1$.

***Solution*:** The line $y = 4x - 1$ intersects the parabola $y^2 = 2x$ at *A* and *B*

$$2x = (4x-1)^2 \Rightarrow 16x^2 - 10x + 1 = 0$$

$$\Rightarrow \quad (8x-1)(2x-1) = 0$$

$$\therefore \quad A = \left(\frac{1}{2}, 1\right) \text{ and } B = \left(\frac{1}{8}, -\frac{1}{2}\right)$$



If the formula $\int y\,dx$ is to be used then the area will have to be split up as *OBC* and *CBA*. Instead the problem can be done directly by using the formula $\int (x_2 - x_1)\,dy$.

$$\text{Area required } = \int_{y=-1/2}^{1}(x_2 - x_1)\,dy = \int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy = \left[\frac{y^2}{8} + \frac{y}{4} - \frac{y^3}{6}\right]_{-1/2}^{1}$$

$$= \left(\frac{1}{8} + \frac{1}{4} - \frac{1}{6}\right) - \left(\frac{1}{32} - \frac{1}{8} + \frac{1}{48}\right)$$

$$= \frac{(3+6-4)}{24} - \frac{(3-12+2)}{96} = \frac{5}{24} + \frac{7}{96} = \frac{27}{96} = \frac{9}{32} \text{ sq. units}$$

EXERCISE – I

1. Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y = 16x$.

2. Using integration, find the area of the region bounded between the line x = 2 and the parabola $y^2 = 8x$.

3. Find the area of the region between the curves $y^2 = 4x$ and x = 3.

4. Make a rough sketch of the graph of the function $y = 4 - x^2, 0 \leq x \leq 2$ and determine the area enclosed by the curve, the x-axis and the lines x = 0 and x = 2.

5. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines x = 2 and x = 8.

6. Find the area of the region bounded by $y = |x - 1|$ and $y = 1$.

7. Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2$ and $x = 3$.

8. Find the area of the region bounded by $y^2 = 9x$, $x = 2, x = 4$ and the x-axis in the first quadrant.

9. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

10. Find the area of the region in the first quadrant enclosed by x-axis, line $= \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

11. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

12. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

13. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

14. Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

15. Sketch the curves $y = e^x; y = xe^x$ between x = –1, and x = 1. Calculate the area enclosed between portions of these two curves and the lines x = – 1, x = 1.

EXERCISE – II

1. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.

2. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and x-axis.

3. Using integration, find the area of the triangular region, the equations of whose sides are $y = 2x + 1,\ y = 3x + 1$ and $x = 4$.

4. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

5. Sketch the graph $y = |x - 5|$. Evaluate $\int_0^1 |x - 5|\,dx$. What does this value of the integral represent on the graph.

6. Sketch the graph $y = |x + 1|$. Evaluate $\int_{-4}^{2} |x + 1|\ dx$. What does the value of this integral represent on the graph?

7. Prove that the area common to the two parabolas $y = 2x^2$ and $y = x^2 + 4$ is $\frac{32}{3}$ sq. units.

8. Find the area bounded by the curve $y = x\,|\,x\,|$, x-axis and the ordinates $x = 1$ and $x = -1$.

9. Find the area bounded by the lines $y = 4x + 5,\ y = 5 - x$ and $4y = x + 5$.

10. Find the area of the parabola $y^2 = 4ax$ bounded by its latus-rectum.

11. Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.

12. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$

13. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

14. Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$.

15. Using the method of integration find area bounded by the curve $|x| + |y| = 1$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\frac{9}{2}$ sq. units

PP2. 18 sq. units

PP3. 9 sq. units

PP4. $\frac{8}{3}a^2$ sq. units

PP5. $\frac{4}{3}$ sq. units

PP6. $\frac{9}{8}$ sq. units

PP7. $\frac{21}{2}$ sq. units

PP8. 5 sq. units

PP9. 6 sq. units

PP10. 4

ANSWERS TO EXERCISE – I

1. $\frac{128}{3}$ sq. units

2. $\frac{32}{3}$ sq. units

3. $8\sqrt{3}$ sq. units

4. $\frac{16}{3}$ sq. units

5. $\frac{\pi a^2}{4}$ sq. units

6. 16 sq. units

8. $16-4\sqrt{2}$

9. 12π

10. $\frac{\pi}{3}$

11. $\frac{9}{8}$

12. $8\sqrt{3}$

13. $\frac{1}{3}$

14. $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$

15. $e-\frac{3}{e}$

ANSWERS TO EXERCISE – II

1. $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units

2. $\left(2 - \sqrt{2}\right)$ sq. units

3. 8 sq. units

4. $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$ sq. units

5. $\frac{9}{2}$ sq. units

6. 9 sq. units

8. $\frac{2}{3}$ sq. units

9. $\frac{15}{2}$ sq. units

10. $\frac{8}{3}a^2$

11. $\frac{13}{3}$

12. 4

14. $\frac{23}{6}$

15. 2