LESSON 7

INDEFINITE & DEFINITE INTEGRATION

In the previous chapter, we have studied about the differentiation and its application. Now in this chapter we will study about another main branch of calculus called integration.

Integration is the inverse process of differentiation. The process of finding $f(x)$, when its derivative $f'(x)$ is given is known as integration.

1. INTEGRAL AS ANTI-DERIVATIVE

If $f(x)$ is a differentiable function such that $f'(x) = g(x)$, then integration of $g(x)$ w.r.t. x is $f(x) + c$. Symbolically it is written as

$$\int g(x)dx = f(x) + c$$

here c is known as constant of integration and it can take any real value. For example $\frac{d}{dx}(\tan x) = \sec^2 x$, so

$$\int \sec^2 x \, dx = \tan x + c$$

2. LIST FOR STANDARD FOMULAE

Based upon the about method and the previous knowledge of differentiation of standard functions, here is the list of integration of standard functions.

Functions $f(x)$ (Integrand)	Integration $\int f(x)\,dx$
constant k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
$1/x \quad (x \neq 0)$	$\ln\lvert x\rvert + c$
$a^x \quad (a > 0)$	$a^x/\ln a + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\text{cosec}^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\text{cosec}\, x \cot x$	$-\text{cosec}\, x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
$1/(1+x^2)$	$\tan^{-1} x + c$
$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1} x + c$

Theorem1.

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

Proof: Let $\int f(x)\,dx = F(x)$ …(i)

Then, $\frac{d}{dx}\{F(x)\} = f(x)$

$\therefore \quad \frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$

Theorem 2.

Two integrals of the same function can differ only by a constant.

Proof:

Let $f_1(x)$ and $f_2(x)$ be two integrals of $g(x)$. Then by definition $f_1'(x) = g(x)$ and $f_2'(x) = g(x)$ for all possible values of real x.

$\Rightarrow \quad f_1'(x) = f_2'(x) \;\; \forall\, x \in R$

Let $h(x) = f_1(x) - f_2(x)$

$\Rightarrow$ $h'(x) = 0 \ \forall\ x \in R$

Now consider the interval $[a, b]$ $(a < b)$, then by Lagrange's Mean value's theorem, there exists some $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

Since $h'(x) = 0 \ \forall\ x \in R$ so $h'(c) = 0$

$\Rightarrow$ $h(b) = h(a)$

$\Rightarrow$ $h(x)$ is a constant function

Let $h(x) = c$

$\Rightarrow$ $f_1(x) - f_2(x) = c$

Hence two integrals of the same function can differ only by a constant.

Theorem 3.

(i) $\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$

where a and b are constants

(ii) $\int f(x)dx = g(x) + c$, then $\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + c$

where a and b are constants and $a \neq 0$

Illustration 1

Question: **Evaluate:** $\int (\sqrt{3}\sin x - \cos x)\,dx$

Solution: $\int (\sqrt{3}\sin x - \cos x)dx \quad = \sqrt{3}\int \sin x\,dx - \int \cos x\,dx$

$$= \quad -\sqrt{3}\cos x - \sin x + c = -\ 2\left[\cos x.\cos\frac{\pi}{6} + \sin x.\sin\frac{\pi}{6}\right] + c = -2\cos\left(x - \frac{\pi}{6}\right) + c$$

Illustration 2

Question: **Evaluate:** $\int \sec^2(3x + 5)\,dx$

Solution: We know that $\int \sec^2 x\,dx = \tan x + c$

so $\int \sec^2(3x+5)\,dx = \frac{1}{3}\tan(3x+5) + c$

3. INTEGRATION BY SUBSTITUTION

It is not always possible to find the integral of a complicated function only by observation, so we need some methods of integration and integration by substitution is one of them. This methods has 3 parts:

(i) Direct substitution

(ii) Standard substitution

(iii) Indirect substitution

3.1 DIRECT SUBSTITUTION

If $\int f(x)\, dx = g(x) + c$ then in $I = \int f(h(x))\, h'(x)\, dx$, we put $h(x) = t \Rightarrow h'(x)\, dx = dt$

so $I = \int f(t) dt = g(t) + c = g(h(x)) + c$

Illustration 3

Question: **Evaluate: $\int \cot x\, dx$**

Solution: $I = \int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x}$ put $\sin x = t$

$\Rightarrow$ $\cos x\, dx = dt$

so $I = \int \frac{dt}{t} = \ln|t| + c = \ln|\sin x| + c$

3.2 STANDARD SUBSTITUTION

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows:

Integrand	**Substitution**
$x^2 + a^2$ or $\sqrt{x^2 + a^2}$	$x = a\tan\theta$
$x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta$
$a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a\cos 2\theta$
$(x \pm \sqrt{x^2 \pm a^2})^n$	expression inside the bracket = t

$\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
$2x^2 - 1$	$x = \cos\theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}$ $(n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

Illustration 4

Question: **Evaluate:** $\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$

Solution: $I = \int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}} \quad = \int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{15/16}(x-4)^2}$

Put $\frac{x+3}{x-4} = t \quad \Rightarrow \left(\frac{(x-4)-(x+3)}{(x-4)^2}\right)dx = dt \quad \Rightarrow \frac{dx}{(x-4)^2} = \frac{dt}{-7}$

So $I = \frac{-1}{7}\int \frac{dt}{t^{15/16}} = \frac{-1}{7}\int t^{-15/16}dt = \frac{-1}{7}\frac{t^{1/16}}{1/16} + c = \frac{-16}{7}t^{1/16} + c = \frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16} + c$

3.3 INDIRECT SUBSTITUTION

If integrand $f(x)$ can be rewritten as product of two functions. $f(x) = f_1(x)\, f_2(x)$, where $f_2(x)$ is a function of integral of $f_1(x)$, then put integral of $f_1(x) = t$.

Illustration 5

Question: **Evaluate:** $\int \sqrt{\frac{x}{4-x^3}}\, dx$

Solution: $I = \int \sqrt{\frac{x}{4-x^3}}\, dx = \int \frac{\sqrt{x}\, dx}{\sqrt{4-x^3}}$

Here integral of $\sqrt{x} = \frac{2}{3}x^{3/2}$ and $4 - x^3 = 4 - (x^{3/2})^2$

Put $x^{3/2} = t \quad \Rightarrow \quad \sqrt{x}\, dx = \frac{2}{3}dt$

So $\quad I = \frac{2}{3}\int \frac{dt}{\sqrt{4-t^2}}$

$$= \frac{2}{3}\sin^{-1}\left(\frac{t}{2}\right) + c \quad = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right) + c$$

4. INTEGRATION BY PARTS

If integrand can be expressed as product of two functions, then we use the following formula.

$$\int f_1(x)\ f_2(x)\ dx = f_1(x)\ \int f_2(x)\,dx - \int ((f_1'(x)\int f_2(x)\ dx))\,dx$$

where $f_1(x)$ and $f_2(x)$ are known as first and second function respectively.

Remarks:

(i) We do not put constant of integration in 1st integral, we put this only once in the end.

(ii) Order of $f_1(x)$ and $f_2(x)$ is normally decided by the rule ILATE, where I $\rightarrow$ inverse, L $\rightarrow$ Logarithms, A $\rightarrow$ Algebric, T $\rightarrow$ trigonometric and E $\rightarrow$ exponential.

Illustration 6

Question: **Evaluate :** $\int x^2 \sin x\ dx$

Solution: $\int x^2 \sin x\ dx$

$$= x^2\int \sin x\,dx - \int (2x\int \sin x\,dx)\,dx = -x^2\cos x + 2\int x\cos x\ dx$$

$$= -x^2\cos x + 2[x\int \cos x\,dx - \int (1\int \cos x\,dx)\ dx = -x^2\cos x + 2[x\sin x - \int \sin x\,dx]$$

$$= -x^2\cos x + 2x\ \sin x - 2\cos x + c$$

4.1 SPECIAL USE OF INTEGRATION BY PARTS

(i) $\int f(x)\ dx = \int (f(x))\,.\,1\,dx$

Now integrate taking $f(x)$ as 1st function and 1 as 2nd function. In most of the cases, $f(x)$ is an inverse or logarithmic function.

(ii) $\int \frac{f(x)}{[g(x)]^n}\ dx = \int \frac{f(x)}{g'(x)}\cdot\frac{g'(x)}{[g(x)]^n}\ dx$

Now integrate taking $\frac{f(x)}{g'(x)}$ as 1st function and $\frac{g'(x)}{[g(x)]^n}$ as 2nd function.

(iii) If integrand is of the form $e^x f(x)$, then rewrite $f(x)$ as sum of two functions in which one is derivative of other.

$$\int e^x f(x)\ dx = \int e^x (g(x) + g'(x))\,dx$$

$$= e^x g(x) + c$$

Illustration 7

Question: **Evaluate:** $\int \ln x\, dx$.

Solution: $I = \int \ln x\, dx = \int (\ln x \,.\, 1)\, dx = \ln x \,.\, x - \int \frac{1}{x} \,.\, x\, dx = x\ln x - x + c = x(\ln x - 1) + c$

5. INTEGRATION USING PARTIAL FRACTIONS

When integrand is a rational function i.e. of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the polynomials functions of x, we use the method of partial fraction. For example we can rewrite $\frac{1}{(3x-1)(3x+2)}$ as $\frac{1}{3(3x-1)} - \frac{1}{3(3x+2)}$.

If the degree of $f(x)$ is less then degree of $g(x)$ and $g(x) = (x-a_1)^{\alpha_1}$ …… $(x^2+b_1x+c_1)^{\beta_1}$ ……. then we can put

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \ldots\ldots + \frac{A_{\alpha_1}}{(x-a_1)^{\alpha_1}} \ldots\ldots\ldots$$

$$+ \frac{B_1x+C_1}{(x^2+b_1x+c_1)} + \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + \ldots\ldots + \frac{B_{\beta_1}x+C_{\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} \ldots\ldots$$

Here A_1, A_2……., A_{α_1}…….. , B_1 , B_2 ……. , B_{β_1}…… C_1, C_2……. C_{β_1} …….. are the real constants and these can be calculated by reducing both sides of the above equation as identity in polynomial form and then by comparing the coefficients of like powers. The constants can also be obtained by putting some suitable numerical values of x in both sides of the identity.

If degree of $f(x)$ is more than or equal to degree of $g(x)$, then divide $f(x)$ by $g(x)$ so that the remainder has degree less than of $g(x)$.

Illustration 8

Question: **Evaluate :** $\int \frac{dx}{(x-1)(x-2)(x-3)}$.

Solution: Put $\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$\Rightarrow \quad 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

Put $x = 1$, we get, $A = \frac{1}{2}$

$x = 2$, we get, $B = -1$

$x = 3$, we get, $C = \frac{1}{2}$

So Integral = $\frac{1}{2}\int \frac{dx}{x-1} - \int \frac{dx}{x-2} + \frac{1}{2}\int \frac{dx}{x-3} = \frac{1}{2}\ln|x-1| - \ln|x-2| + \frac{1}{2}\ln|x-3| + c$

$$= \ln\left(\frac{\sqrt{x^2-4x+3}}{|x-2|}\right) + c$$

Illustration 9

Question: **Evaluate :** $\int \frac{dx}{(x+2)(x^2+1)}$.

Solution: Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{(x^2+1)}$

$\Rightarrow$ $1 = A(x^2+1) + (Bx+C)(x+2)$ Put $x = -2$, we get $A = \frac{1}{5}$

Now compare the coefficients of x^2 and constant term we get

$0 = A + B$ and $1 = A + 2C$

$\Rightarrow$ $B = -\frac{1}{5}$, $C = \frac{2}{5}$

So $I = \frac{1}{5}\int \frac{dx}{x+2} - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{2}{5}\int \frac{dx}{x^2+1}$

$$= \frac{1}{5}\ln|x+2| - \frac{1}{10}\ln(x^2+1) + \frac{2}{5}\tan^{-1}x + C .$$

6. INTEGRATION USING TRIGONOMETRIC IDENTITIES

When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can easily be integrated. Some of the identities useful for this purpose are given below:

(i) $2\sin^2\left(\frac{x}{2}\right) = (1-\cos x)$

(ii) $2\cos^2\left(\frac{x}{2}\right) = (1+\cos x)$

(iii) $2\sin a\cos b = \sin(a+b) + \sin(a-b)$

(iv) $2\cos a\sin b = \sin(a+b) - \sin(a-b)$

(v) $2\cos a\cos b = \cos(a+b) + \cos(a-b)$

(vi) $2\sin a\sin b = \cos(a-b) - \cos(a+b)$

Illustration 10

Question: **Evaluate $\int \sin 3x \sin 2x \, dx$.**

Solution: Using $2\sin a \sin b = \cos(a-b) - \cos(a+b)$, we have

$$\int \sin 3x \sin 2x \, dx = \frac{1}{2}\int 2\sin 3x \sin 2x \, dx$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\int \cos x \, dx - \frac{1}{2}\int \cos 5x \, dx = \frac{1}{2}\sin x - \frac{\sin 5x}{10} + c$$

7. ALGEBRAIC INTEGRALS

Using the technique of standard substitution and integration by parts, we can derive the following formulae.

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

7.1 INTEGRAL OF THE FORM

$$\int \frac{dx}{ax^2 + bx + c}\ ,\ \int \frac{dx}{\sqrt{ax^2 + bx + c}}\ ,\ \int \sqrt{ax^2 + bx + c}\ dx$$

Here in each case write $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ put $x + \frac{b}{2a} = t$ and use the standard formulae.

Illustration 11

Question: **Evaluate :** $\int \frac{dx}{\sqrt{-x^2 + 4x + 6}}$.

Solution: $-x^2 + 4x + 6 = -(x^2 - 4x + 4) + 10 = 10 - (x - 2)^2$

$$I = \int \frac{dx}{\sqrt{10 - (x-2)^2}} \qquad \text{put } x - 2 = t \Rightarrow dx = dt$$

$$I = \int \frac{dt}{\sqrt{10 - t^2}} = \sin^{-1}\frac{t}{\sqrt{10}} + c = \sin^{-1}\left(\frac{x-2}{\sqrt{10}}\right) + c$$

Illustration 12

Question: **Evaluate:** $\int \sqrt{3x^2 - 6x + 10}\ dx$.

Solution: $3x^2 - 6x + 10 = 3(x - 1)^2 + 7$ Put $x - 1 = t$

$\Rightarrow$ $dx = dt$

$$I = \sqrt{3}\int \sqrt{t^2 + \frac{7}{3}}\ dt = \sqrt{3}\left[\frac{t}{2}\sqrt{t^2 + \frac{7}{3}} + \frac{7}{6}\ \ln\left|t + \sqrt{t^2 + \frac{7}{3}}\right|\right] + c \text{ ,where } t = x - 1$$

7.2 INTEGRALS OF THE FORM

$$\int \frac{(ax+b)\,dx}{\sqrt{cx^2+ex+f}}\ ,\ \int \frac{(ax+b)\,dx}{cx^2+ex+f}\ ,\ \int (ax+b)\sqrt{cx^2+ex+f}\ dx$$

Here write $ax + b = A(2cx + e) + B$

Find A and B by comparing, the coefficients of x and constant term.

Illustration 13

Question: **Evaluate :** $\int \frac{(3x+5)\,dx}{\sqrt{x^2+4x+3}}$.

Solution: Write $3x + 5 = A(2x + 4) + B \quad \Rightarrow \quad A = \frac{3}{2}, B = -1$

So $I = \frac{3}{2}\int \frac{2x+4}{\sqrt{x^2+4x+3}} - \int \frac{dx}{\sqrt{x^2+4x+3}}$

In 1st integral put $x^2 + 4x + 3 = t \Rightarrow \quad (2x + 4)\,dx = dt$

$$I = \frac{3}{2}\int \frac{dt}{\sqrt{t}} - \int \frac{dx}{\sqrt{(x+2)^2-1}} \qquad = 3\sqrt{x^2+4x+3} - \ln\left|(x+2)+\sqrt{x^2+4x+3}\right| + c$$

7.3 INTEGRALS OF THE FORM

$$\int \frac{(ax^2+bx+c)\,dx}{\sqrt{(ex^2+fx+g)}}\ ,\ \int \frac{(ax^2+bx+c)\,dx}{(ex^2+fx+g)}\ ,\ \int (ax^2+bx+c)\sqrt{(ex^2+fx+g)}\,dx$$

Here put $ax^2 + bx + c = A(ex^2 + fx + g) + B(2ex + f) + c$

find the values of A, B and C by comparing the coefficient of x^2, x and constant term.

Illustration 14

Question: $\int \frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}$.

Solution: Let $x^2 + 4x + 7 = A(x^2 + x + 1) + B(2x + 1) + C$

Comparing the coefficients of x^2, x and constant term, we get

$$A = 1,\ A + 2B = 4,\ A + B + C = 7 \qquad \Rightarrow \qquad A = 1,\ B = \frac{3}{2},\ C = \frac{9}{2}$$

So $I = \int \sqrt{x^2+x+1}\,dx + \frac{3}{2}\int \frac{(2x+1)\,dx}{\sqrt{x^2+x+1}} + \frac{9}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$

Now $\quad x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

So $\quad I = \left(\frac{x + \frac{1}{2}}{2}\right)\sqrt{x^2 + x + 1} + \frac{3}{8}\ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right)$

$$+ 3\sqrt{x^2 + x + 1} + \frac{9}{2}\ln\left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + c$$

7.5 INTEGRALS OF THE FORM

$$\int \frac{dx}{(ax + b)\sqrt{ex^2 + fx + g}}$$

Here put $ax + b = \frac{1}{t}$

Illustration 15

Question: **Evaluate :** $\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}}$.

Solution: Put $\quad x + 2 = \frac{1}{t} \quad \Rightarrow \quad dx = \frac{-dt}{t^2}$

Now $\quad x^2 + 4x + 8 = (x + 2)^2 + 4$

So $\quad I = \int \frac{-dt}{t\sqrt{\frac{1}{t^2} + 4}} = -\int \frac{dt}{\sqrt{1 + 4t^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 + \frac{1}{4}}} = -\frac{1}{2}\ln\left|t + \sqrt{t^2 + \frac{1}{4}}\right| + c$

$$= -\frac{1}{2}\ln\left|\frac{1}{x + 2} + \sqrt{\frac{1}{(x + 2)^2} + \frac{1}{4}}\right| + c$$

7.6 INTEGRALS OF THE FORM

$$\int \frac{(ax + b)\,dx}{(cx + e)\sqrt{ex^2 + fx + g}}$$

Here put $(ax + b) = A\,(cx + e) + B$

Find the values of A and B by comparing the coefficients of x and constant term.

Illustration 16

Question: **Evaluate :** $\int \frac{(4x+7)}{(x+2)\sqrt{x^2+4x+8}}$.

Solution: Let $4x + 7 = A(x + 2) + B$

$\Rightarrow$ $A = 4, B = -1$

So $I = 4\int \frac{dx}{\sqrt{x^2+4x+8}} - \int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$= 4\ln\left(x+2+\sqrt{x^2+4x+8}\right) + \frac{1}{2}\ln\left|\frac{1}{x+2} + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}}\right| + c$

7.7 INTEGRALS OF THE FORM

$$\int \frac{(ax^2+bx+c)\,dx}{(ex+f)\sqrt{gx^2+hx+i}}$$

Here put $ax^2 + bx + c = A(ex + f)(2gx + h) + B(ex + f) + C$

Find the values of A , B and C by comparing the coefficients of x^2, x and constant term.

Illustration 17

Question: **Evaluate :** $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$.

Solution: Put $2x^2 + 7x + 11 = A(x + 2)(2x + 4) + B(x + 2) + C$

Compare the coefficient of x^2, x and constant term, we get

$A = 1,\ 7 = 8A + B,\ C + 2B + 8A = 11 \Rightarrow B = -1, C = 5$

So $I = \int \frac{2x+4}{\sqrt{x^2+4x+8}} - \int \frac{dx}{\sqrt{x^2+4x+8}} + 5\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}}$

$= 2\sqrt{x^2+4x+8} - \ln\left|(x+2)+\sqrt{x^2+4x+8}\right|$

$-\frac{5}{2}\ln\left|\frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2}+\frac{1}{4}}\right| + c$

7.8 STANDARD SUBSTITUTION

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows:

Integrand	**Substitution**
$x^2 + a^2$ or $\sqrt{x^2+a^2}$	$x = a\tan\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a\sec\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a\cos 2\theta$
$(x \pm \sqrt{x^2 \pm a^2})^n$	expression inside the bracket = t
$\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
$2x^2 - 1$	$x = \cos\theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}$ $(n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

Illustration 18

Question: **Evaluate:** $\displaystyle\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$

Solution: $I = \displaystyle\int \frac{dx}{(x+3)^{15/16}(x-4)^{17/16}}$

$$= \int \frac{dx}{\left(\frac{x+3}{x-4}\right)^{15/16}(x-4)^2}$$

Put $\frac{x+3}{x-4} = t \Rightarrow \left(\frac{(x-4)-(x+3)}{(x-4)^2}\right)dx = dt \Rightarrow \frac{dx}{(x-4)^2} = \frac{dt}{-7}$

So $I = \frac{-1}{7}\displaystyle\int \frac{dt}{t^{15/16}} = \frac{-1}{7}\int t^{-15/16}dt$

$$= \frac{-1}{7}\frac{t^{1/16}}{1/16} + c = \frac{-16}{7}t^{1/16} + c = \frac{-16}{7}\left(\frac{x+3}{x-4}\right)^{1/16} + c$$

8. GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

Let $f(x)$ be a function defined on a closed interval $[a, b]$. Then $\int_a^b f(x)\,dx$ represents the algebraic sum of the areas of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$, $x = b$. Here algebraic sum means that area which is above the x-axis will be added in this sum with + sign and area which is below the x-axis will be added in this sum with – sign. So value of the definite integral may be positive, zero or negative.

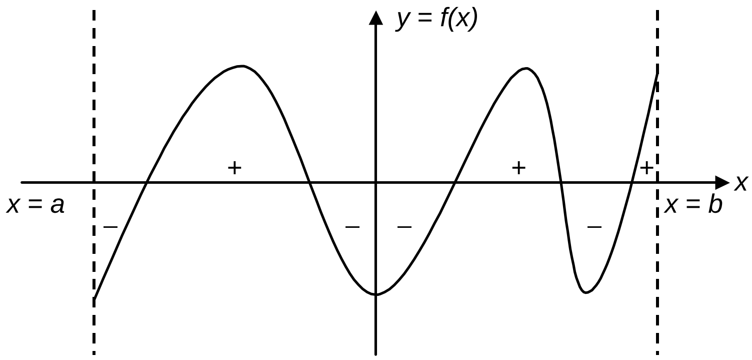
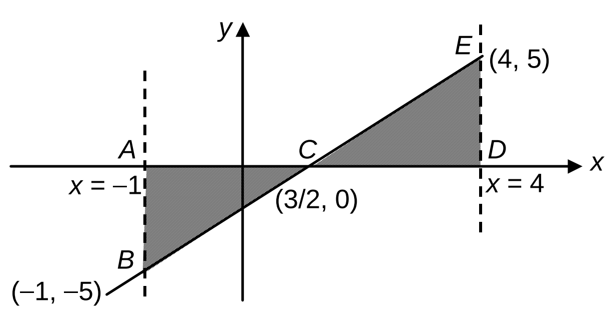


Illustration 19

Question: **Evaluate:** $\int_{J1}^{4} (2x \text{ J } 3)\,dx$

Solution: $y = 2x - 3$ is a straight line, which lie below the x-axis in $\left[-1, \frac{3}{2}\right)$ and above in $\left(\frac{3}{2}, 4\right]$



Now area of $\Delta ABC = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

Area of $\Delta CDE = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

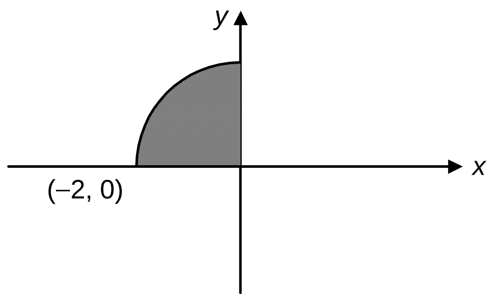
So $\int_{-1}^{4} (2x - 3)\,dx = -\frac{25}{4} + \frac{25}{4} = 0$

Illustration 20

Question: **Evaluate:** $\int_{-2}^{0} \sqrt{4 - x^2}\, dx$

Solution: $y = \sqrt{4 - x^2}$, $x \in [-2, 0]$

Represents a quarter circle in 2nd quadrant, which is above the *x*-axis radius of circle is 2.

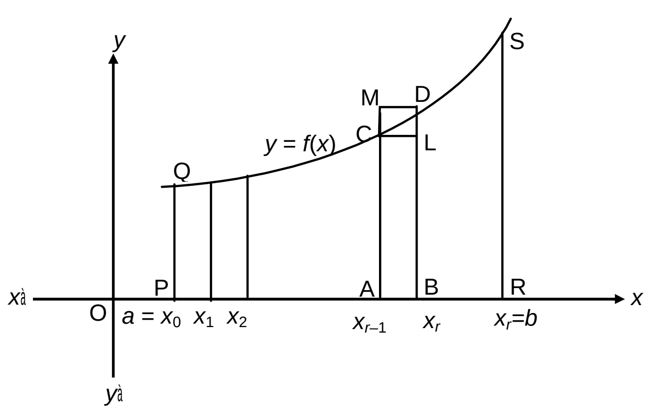


so $\int_{-2}^{0}\left(\sqrt{4 - x^2}\right) dx = \frac{1}{4}\left[\pi (2)^2\right] = \pi$ square unit

9. DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let *f* be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non-negative, so the graph of the function is a curve above the *x*-axis.

The definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the *x*-axis. To evaluate this area, consider the region *PRSQP* between this curve, *x*-axis and the ordinates $x = a$ and $x = b$.



Divide the interval $[a, b]$ into *n* equal subintervals denotes by

$[x_0, x_1], [x_1, x_2], ..., [x_{r-1}, x_r], ..., [x_{n-1}, x_n]$, where

$x_0 = a, x_1 = a + h, x_2 = a + 2h, ..., x_r = a + rh$ and $x_n = b = a + nh$ or $n = \frac{b-a}{h}$.

We note that as $n \to \infty, h \to 0$

The region $PRSQP$ under consideration is the sum of n sub regions, where each sub-region is defined on subintervals $[x_{r-1}, x_r], r = 1, 2, 3, ..., n$.

Given the figure, area of the rectangle ($ABLC$) < area of the region ($ABDCA$) < area of the rectangle ($ABDM$) …(i)

Evidently as $x_r - x_{r-1} \to 0$ i.e., $h \to 0$ all the three areas shown in (i) become nearly equal to each other. Now we form the following sums.

$$s_n = h[f(x_0) + ... + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad \text{…(ii)}$$

and $S_n = h[f(x_1) + f(x_2) + ... + f(x_n)] = h\sum_{r=1}^{n} f(x_r)$ ….(iii)

Here s_n and S_n denote the sum of area of all lower rectangle and upper rectangle raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, ..., n$ respectively.

In view of the inequality (i) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$s_n <$ area of the region $PRSQP < S_n$ …(iv)

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values (ii) and (iii) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n$ = area of the region $PRSQP$ = $\int_a^b f(x)\,dx$ …(v)

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangle above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (v) as

$$\int_a^b f(x) = dx = \lim_{h\to 0} h[f(a) + f(a+h) + ... + f(a + (n-1)h)]$$

or $\int_a^b f(x) = dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + ... + f(a + (n-1)h)]$ …(vi)

where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

The above expression (vi) is known as the definition of definite integral as the limit of sum.

Illustration 21

Question: **Find $\int_0^2 \left(x^2+1\right)dx$ as the limit of a sum.**

Solution: By definition

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+.....+f(a+(n-1)h)],$$

where $h=\frac{b-a}{n}$

In this example $a=0,\ b=2,\ f(x)=x^2+1,\ h=\frac{2-0}{n}=\frac{2}{n}$

Therefore, $\int_0^2 \left(x^2+1\right)dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+.....+f\left(\frac{2(n-1)}{n}\right)\right]$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{2^2}{n^2}+1\right)+\left(\frac{4^2}{n^2}+1\right)+......+\left(\frac{(2n-2)^2}{n^2}+1\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}[(\underbrace{1+1+......+1}_{n\text{ terms}})+\frac{1}{n^2}\left(2^2+4^2+.....+(2n-2)^2\right)]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}\left(1^2+2^2+.....+(n-1)^2\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{4}{n^2}\frac{(n-1)n(2n-1)}{6}\right]=2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2}{3}\frac{(n-1)(2n-1)}{n}\right]$$

$$=2\lim_{n\to\infty}\left[1+\frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]=2\left[1+\frac{4}{3}\right]=\frac{14}{3}$$

10. GENERAL PROPERTIES OF DEFINITE INTEGRAL

10.1 $\int_a^b f(x)dx = \int_a^b f(t)\,dt = \int_a^b f(y)dy$

i.e. variable of integration in definite integral is *a* dummy variable.

Illustration 22

Question: **If $f(x) = \int_1^x \left(\frac{\ln t}{1+t}\right) dt$ $(x > 0)$, prove that $f(x) + f\left(\frac{1}{x}\right) = \frac{(\ln x)^2}{2}$**

Solution: $f(x) = \int_1^x \left(\frac{\ln t}{1+t}\right) dt \Rightarrow f\left(\frac{1}{x}\right) = \int_1^{1/x} \left(\frac{\ln t}{1+t}\right) dt$

Now to add $f(x)$ and $f\left(\frac{1}{x}\right)$, we have to make both the limits of integration same (by 3.2)

Put $\frac{1}{t} = y \Rightarrow \frac{dt}{-t^2} = dy \Rightarrow dt = -\frac{dy}{y^2}$

So $\int_1^{1/x} \left(\frac{\ln t}{1+t}\right) dt = \int_1^x \frac{\ln(1/y)}{1+\frac{1}{y}} \left(\frac{-dy}{y^2}\right) = \int_1^x \frac{\ln y}{y(1+y)} dy = \int_1^x \frac{\ln t}{t(1+t)} dt$ (by 3.3)

Hence $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left(\frac{\ln t}{1+t} + \frac{\ln t}{t(1+t)}\right) dt = \int_1^x \frac{\ln t}{1+t}\left(1+\frac{1}{t}\right) dt$

$$= \int_1^x \frac{\ln t}{t} dt = \left.\frac{(\ln t)^2}{2}\right|_1^x = \frac{(\ln x)^2}{2}$$

10.2 $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

Illustration 23

Question: **Evaluate $\int_2^3 \frac{dx}{x\sqrt{4x^2 - 1}}$**

Solution: $I=\int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$ Put $x=\frac{1}{t} \Rightarrow dx=-\frac{dt}{t^2}$

So $I=\int_{1/2}^{1/3} \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\frac{4}{t^2}+1}}=-\int_{1/2}^{1/3}\frac{dt}{\sqrt{4+t^2}}$

$=\int_{1/3}^{1/2}\frac{dt}{\sqrt{4+t^2}}=\ln\left(t+\sqrt{4+t^2}\right)\Big|_{1/3}^{1/2}=\ln\left(\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right)$

10.3 $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

Illustration 24

Question: **Evaluate** $\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$

Solution: $I=\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$ …(i)

$I=\int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{9-(9-x)}}\,dx \Rightarrow I=\int_2^7\frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}\,dx$ …(ii)

adding (i) and (ii), we get, $2I=\int_2^7\left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}+\frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\right)dx$

$=\int_2^7 dx = x\Big|_2^7=7-2=5$

so $I=\frac{5}{2}$

11. PROPERTIES OF DEFINITE INTEGRALS

11.1 Property-I

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ **i.e., Integration is independent of the change of variable**

Proof: Let $\phi(x)$ be a primitive of $f(x)$. Then

$$\frac{d}{dx}\{\phi(x)\} = f(x) \Rightarrow \frac{d}{dt}\{\phi(t)\} = f(t)$$

Hence $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$ ...(i)

and $\int_a^b f(t)dt = [\phi(t)]_a^b = \phi(b) - \phi(a)$ ...(ii)

From (i) and (ii), we have $\int_a^b f(x)dx = \int_b^a f(t)dt$

11.2 Property-II

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ **i.e., if the limits of a definite integral are interchanged then its value changes by minus sign only.**

Proof: Let $\phi(x)$ be a primitive of $f(x)$. Then

$$\int_a^b f(x)dx = \phi(b) - \phi(a)$$

and $-\int_b^a f(x)dx = -[(\phi(a) - \phi(b))] = \phi(b) - \phi(a)$

$$\therefore \int_a^b f(x)dx = -\int_b^a f(x)dx$$

11.3 Property-III

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{, where } a < c < b.$$

Proof: Let $\phi(x)$ be a primitive of $f(x)$. Then

$$\int_a^b f(x)dx = \phi(b) - \phi(a) \qquad \ldots\text{(i)}$$

and $\int_a^c f(x)dx + \int_c^b f(x)dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \qquad \ldots\text{(ii)}$

From (i) and (ii), we get $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Generalization: The above property can be generalized into the following form:

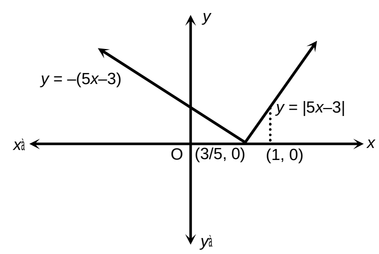
$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots + \int_a^b f(x)dx \text{, where } a < c_1 < c_2 < c_3 \ldots\ldots < c_{n-1} < c_n < b$$

Illustration 25

Question: **Evaluate $\int_0^1 |5x-3|\,dx$.**

Solution:

$$|5x-3| = \begin{cases} -(5x-3) \text{ when } 5x-3<0 \text{ i.e., } x < \frac{3}{5} \\ 5x-3 \text{ when } 5x-3 \ge 0 \text{ i.e., } x \ge \frac{3}{5} \end{cases}$$

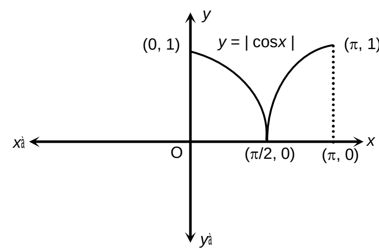


$$\therefore \int_0^1 |5x-3|\,dx = \int_0^{3/5} |5x-3|dx + \int_{3/5}^1 |5x-3|dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^1 = \left(\frac{9}{5} - \frac{9}{10}\right) + \left(-\frac{1}{2} + \frac{9}{10}\right) = \frac{13}{10}$$

Illustration 26

Question: **Evaluate $\int_0^{\pi} |\cos x|\,dx$.**



Solution: $|\cos x| = \begin{cases} \cos x \text{ when } 0 \le x \le \dfrac{\pi}{2} \\ -\cos x \text{ when } \dfrac{\pi}{2} \le x \le \pi \end{cases}$

$$\therefore \int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/4} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx = [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 + 1 = 2$$

11.4 Property-IV

If $f(x)$ is a continuous function defined on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Proof: Let $x = a + b - t$. Then $dx = -dt$

Also, $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$

$$\therefore \int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^b f(a+b-t)\,dt \Rightarrow \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Hence $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Illustration 27

Question: **Evaluate** $\int_{\pi/6}^{\pi/3} \dfrac{1}{1+\sqrt{\cot x}}\,dx$.

Solution: Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}} dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ …(i)

Then $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$

$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ …(ii)

Adding (i) and (ii), we get

$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$

$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1.\,dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \quad \Rightarrow \quad I = \frac{\pi}{12}$

11.5 Property-V

If $f(x)$ is a continuous function defined on [0, *a*], then

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Proof: Let $x = a - t$. Then $dx = d(a-t) \Rightarrow dx = -dt$

Also, $x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 0$

$\therefore \quad \int_0^a f(x)dx = -\int_a^0 f(a-t)dt$

$\Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-t)dt \quad \Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$

Hence $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Illustration 28

Question: **Prove that** $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$.

Solution: Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$. Then

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 \Rightarrow \quad I = \frac{\pi}{4}$$

11.6 Property-VI

If $f(x)$ is a continuous function defined on [–*a, a*], then

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0,}^{a} f(x)dx & , \text{ if } f(x) \text{ is an even function} \\ 0 & , \text{ if } f(x) \text{ is an odd function} \end{cases}$$

Proof: We have, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

Let $x = -t$. Then $dx = -dt$

Also $x = -a \Rightarrow t = a$ and $x = 0 \Rightarrow t = 0$

$$\therefore \quad \int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)\,dt$$

$$\Rightarrow \quad \int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-t)\,dt \quad \Rightarrow \quad \int_{-a}^{0} f(x) = \int_{0}^{a} f(-x)dx$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)dx$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)dx = \int_{0}^{a} \{f(-x) + f(x)\}dx \quad \Rightarrow \quad \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0,}^{a} f(x)dx & , \quad \text{if } f(-x) = f(x) \\ 0 & , \quad \text{if } f(-x) = -f(x) \end{cases}$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0,}^{a} f(x)dx & , \quad \text{if } f(x) \text{ is an even function} \\ 0 & , \quad \text{if } f(x) \text{ is an odd function} \end{cases}$$

***Illustration* 29**

Question: **Evaluate** $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$ **.**

Solution: Let $f(x) = \sin^2 x$. Then

$$f(-x) = \sin^2(-x) = \{\sin(-x)\}^2 = \{-\sin x\}^2 = \sin^2 x = f(x)$$

$\therefore$ $f(x)$ is an even function

$$\Rightarrow \quad \int_{-\pi/2}^{\pi/2} f(x)dx = 2\int_{0}^{\pi/2} f(x)\,dx \quad \Rightarrow \quad \int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = 2\int_{0}^{\pi/2} \sin^2 x dx$$

Let $I = \int_{0}^{\pi/2} \sin^2 x dx$, then

$$I=\int_0^{\pi/2}\sin^2\left(\frac{\pi}{2}-x\right)dx \;\Rightarrow\; I=\int_0^{\pi/2}\cos^2 x dx$$

$$\therefore\;\; I+I=\int_0^{\pi/2}\sin^2 x dx=\int_0^{\pi/2}\cos^2 x dx$$

$$\Rightarrow\;\; 2I=\int_0^{\pi/2}\left(\sin^2 x+\cos^2 x\right)dx=\int_0^{\pi/2}1\,dx=\frac{\pi}{2}$$

$$\Rightarrow\;\; I=\frac{\pi}{4} \;\;\Rightarrow\;\; \int_0^{\pi/2}\sin^2 x dx=\frac{\pi}{4}$$

Substituting this value in (i), we get $\int_{-\pi/2}^{\pi/2}\sin^2 x\;\, dx=2\times\frac{\pi}{4}=\frac{\pi}{2}$

PRACTICE PROBLEMS

PP1. $\dfrac{\sin x - \cos x}{\sin x + \cos x}$

PP2. $\dfrac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+3}$

PP3. $\displaystyle\int \frac{dx}{x^{3/2}\left(1+\frac{1}{\sqrt{x}}\right)}$

PP4. $\dfrac{2\tan x \sec^2 x}{\tan^2 x + 3\tan x + 2}$

PP5. $\tan^{-1}x$

PP6. $e^x(\cos x - \sin x)$

PP7. $\dfrac{1}{\sqrt{x^2+4x+5}}$

PP8. $\dfrac{2x-3}{\sqrt{x^2+x+1}}$

PP9. $\dfrac{1}{x\sqrt{x^2+x+1}}$

PP10. $\dfrac{1}{1+2\cos x}$

PP11. $\dfrac{1}{\sin^2 x + \sin 2x}$

PP12. $\dfrac{1}{\cos x\sqrt{\sin^2 x - \cos^2 x}}$

PP13. Evaluate : $\displaystyle\int_2^3 \frac{dx}{x^2}$

PP14. Evaluate : $\displaystyle\int_0^{\pi/2} \sqrt{\sin x}\,\cos^5 x\, dx$

PP15. Evaluate : $\displaystyle\int_{-5}^{5} |x+2|\, dx$

PP16. Evaluate : $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\tan^5 x}$

PP17. Evaluate : $\displaystyle\int_0^1 x(1-x)^n\, dx$

PP18. Evaluate : $\displaystyle\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x}$

PP19. Evaluate : $\displaystyle\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Evaluate $\int \frac{dx}{x^4(x^3+2)^3}$.

Solution:

$$I = \int \frac{dx}{x^4(x^3+2)^3}$$

$$= \int \frac{dx}{x^{13}\left(1+\frac{2}{x^3}\right)^3}$$

Put $1+\frac{2}{x^3} = t \quad \Rightarrow \quad -\frac{6}{x^4}dx = dt$

$$I = -\frac{1}{6}\int\left(\frac{1}{x^3}\right)^3 \frac{1}{\left(1+\frac{2}{x^3}\right)^3} - \frac{6dx}{x^4} = -\frac{1}{6}\int\left(\frac{t-1}{2}\right)^3 \frac{1}{t^3}\,dt$$

$$= -\frac{1}{48}\int \frac{(t^3-3t^2+3t-1)}{t^3}\,dt = -\frac{1}{48}\int\left(1-\frac{3}{t}+\frac{3}{t^2}+\frac{1}{t^3}\right)dt$$

$$= -\frac{1}{48}\left[t-3\ln|t|-\frac{3}{t}-\frac{1}{2t^2}\right]+C$$

where $t = 1+\frac{2}{x^3}$

***Example* 2:**

Evaluate $\int \frac{2\,dx}{(1-x)(1+x^2)}$.

Solution:

The integrand $\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{x^2+1}$

$$= \frac{1}{1-x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1}$$

Hence $\int \frac{2\,dx}{(1-x)(1+x^2)} = \int \frac{dx}{1-x} + \frac{1}{2}\int\frac{2x\,dx}{x^2+1} + \int\frac{dx}{x^2+1}$

$$= -\log(1-x) + \frac{1}{2}\log(x^2+1) + \tan^{-1} x$$

***Example* 3:**

Evaluate $\displaystyle \int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx$.

Solution:

Let $z = \log_e x \quad \therefore dz = \frac{1}{x}dx = \frac{dx}{e^z}$

Now, $I = \displaystyle\int\left(\frac{1}{z} - \frac{1}{z^2}\right)e^z\, dz = \int e^z\left(\frac{1}{z} - \frac{1}{z^2}\right)dz$

$= e^z\left(\frac{1}{z}\right)$ since the integrand is of the form $\int e^x\,(f(x) + f'(x))$ dx

$= \dfrac{x}{\log_e x}$

***Example* 4:**

Evaluate $\displaystyle \int \left(\sqrt{\cot x} + \sqrt{\tan x}\right) dx$**.**

Solution:

$I = \displaystyle\int\left(\sqrt{\cot x} + \sqrt{\tan x}\right)dx = \int \frac{\cos x + \sin x}{\sqrt{\sin x\cos x}}\,dx$

$= \sqrt{2}\displaystyle\int \frac{\cos x + \sin x}{\sqrt{2\sin x\cos x}}\,dx \quad = \sqrt{2}\int \frac{(\cos x + \sin x)}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$

$= \sqrt{2}\displaystyle\int \frac{dz}{\sqrt{1-z^2}}$ *where* $z = \sin x - \cos x$

$= \sqrt{2}\sin^{-1} z \quad = \sqrt{2}\sin^{-1}(\sin x - \cos x)$

***Example* 5:**

Find $\displaystyle \int \frac{x^2 - 3}{x^3 - 2x^2 - x + 2}\,dx$**.**

Solution:

Note that $(x^3 - 2x^2 - x + 2) = (x+1)(x-1)(x-2)$

$\therefore \quad \dfrac{x^2-3}{(x+1)(x-1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$

where $A = \dfrac{(-1)^2 - 3}{((-1)-(1))(-1-2)} = \dfrac{-2}{(-2)(-3)} = -\dfrac{1}{3}$

$B = \dfrac{1^2 - 3}{(1+1)(1-2)} = \dfrac{-2}{2(-1)} = 1$

$C = \dfrac{2^2 - 3}{(2+1)(2-1)} = \dfrac{1}{3}$

(**Note:** The constants are obtained respectively by putting

$x = -1$, in all the terms except $x + 1$

$x = 1$, in all the terms except $x - 1$

$x = 2$, in all the terms except $x - 2$).

$$\int \frac{(x^2-3)dx}{(x+1)(x-1)(x-2)} = \int \frac{A}{x+1}dx + \int \frac{B}{x-1}dx + \int \frac{C}{x-2}dx$$

$$= -\frac{1}{3}\, ln\,(x+1) + ln\,(x-1) + \frac{1}{3}\, ln\,(x-2) + C$$

$$= \frac{1}{3}\, ln \left\{\frac{A\,(x-2)\,(x-1)^3}{(x+1)}\right\} \text{ where } C = \frac{1}{3}\, ln\, A$$

***Example* 6:**

Evaluate $\int_0^{\pi/2} \ln(\sin x)\, dx$

Solution:

Let $I = \int_0^{\pi/2} \ln(\sin x)\, dx$

$\Rightarrow \int_0^{\pi/2} \ln\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$

$= \int_0^{\pi/2} \ln(\cos x)\, dx$

Adding both, we get

$$2I = \int_0^{\pi/2} (\ln(\sin x) + \ln(\cos x))dx \quad = \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x)\, dx - \int_0^{\pi/2} \ln 2\, dx \quad = I_1 - \frac{\pi}{2}\ln 2$$

$$I_1 = \int_0^{\pi/2} \ln(\sin 2x)\, dx$$

Put $\quad 2x = t \;\Rightarrow\; dx = \frac{dt}{2}$

$$I_1 = \frac{1}{2}\int_0^{\pi} \ln(\sin t)\, dt$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \ln(\sin t)\, dt + \int_0^{\pi/2} \ln(\sin(\pi - t)\, dt\right] = \int_0^{\pi/2} \ln(\sin t)\, dt$$

$$= \int_0^{\pi/2} \ln(\sin x)\, dx = I$$

So $\; 2I = I - \frac{\pi}{2}\ln 2 \quad \Rightarrow I = \frac{-\pi}{2}\ln 2 = \frac{\pi}{2}\ln\left(\frac{1}{2}\right)$

***Example* 7:**

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \quad a, b > 0$

Solution:

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x} \quad \text{(Dividing Nr and Dr by } \cos^2 x\text{)}$$

Put $\tan x = t \therefore \sec^2 x\, dx = dt$

Given integral $= \displaystyle\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2}\int_0^{\infty} \frac{dt}{\frac{a^2}{b^2} + t^2}$

$$= \frac{1}{b^2}\,\frac{1}{\left(\frac{a}{b}\right)}\left[\tan^{-1}\frac{tb}{a}\right]_0^{\infty} = \frac{1}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2ab}$$

***Example* 8:**

Evaluate $\displaystyle\int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Solution:

Put $\quad x = \alpha\cos^2\theta + \beta\sin^2\theta$

Then $dx = (-2\alpha\cos\theta\sin\theta + 2\beta\sin\theta\cos\theta)\,d\theta$

$x = \alpha \Rightarrow \alpha = \alpha\cos^2\theta + \beta\sin^2\theta \Rightarrow (\alpha-\beta)\sin^2\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x = \beta \Rightarrow \beta = \alpha\cos^2\theta + \beta\sin^2\theta \Rightarrow (\beta-\alpha)\cos^2\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int_{\theta=0}^{\theta=\pi/2} \frac{(2\beta-2\alpha)\sin\theta\cos\theta\,d\theta}{\sqrt{(\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)\,(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}$$

$$= \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta\,d\theta}{\sqrt{(\beta-\alpha)\sin^2\theta\cdot(\beta-\alpha)\cos^2\theta}}$$

$$= \int_0^{\pi/2} 2\,d\theta = [2\theta]_0^{\pi/2} = \pi$$

***Example* 9:**

Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

Solution:

$\frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x \;\therefore$ express the denominator in terms of $(\sin x - \cos x)$ so

$9 + 16\sin 2x = 9 + 32\sin x\cos x = 25 - 16(\sin x - \cos x)^2$

Put $z = \sin x - \cos x \qquad \therefore dz = (\cos x + \sin x)\,dx$

$x = 0,\ z = 0 - 1 = -1$ and $x = \frac{\pi}{4} \Rightarrow z = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$\therefore I = \int_0^{\pi/2} \frac{(\sin x + \cos x)\,dx}{9 + 16\sin 2x}$$

$$= \int_{z=-1}^{z=0} \frac{dz}{25 - 16z^2} = \frac{1}{16}\int_{z=-1}^{z=0} \frac{dz}{(25/16) - z^2}$$

$$= \frac{1}{16} \times \frac{1}{2(5/4)} \left[\log \left| \frac{(5/4)+z}{(5/4)-z} \right| \right]_{-1}^{0}$$

$$= \frac{1}{40} \left[\log 1 - \log \frac{1}{9} \right]$$

$$= \frac{1}{40} \log 9 = \frac{1}{20} \log 3$$

***Example* 10:**

Find the value of $\int_0^{®} \frac{x \sin x}{2 \text{ J } \sin^2 x} dx$.

Solution:

$$I = \int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x} \, dx = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{2 - \sin^2 (\pi - x)} dx, \text{ (using } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx)$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{2 - \sin^2 x} dx$$

$$\therefore \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{-d(\cos x)}{1 + \cos^2 x} = \left[-\frac{\pi}{2} \tan^{-1}(\cos x) \right]_0^{\pi}$$

$$= \left(\frac{-\pi}{2} \right) \left(\frac{-\pi}{4} \right) + \frac{\pi}{2} \left(\frac{\pi}{4} \right) = \frac{\pi^2}{4}$$

EXERCISE – I

1. Evaluate $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$.

2. Evaluate $\int \frac{dx}{\sin x + \sqrt{3} \cos x}$.

3. Evaluate $\int \frac{\sin 2x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$.

4. Evaluate $\int \frac{3^x \, dx}{\sqrt{1-9^x}}$.

5. Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$.

6. Evaluate $\int \frac{\tan(\log x)}{x} dx$.

7. Evaluate $\int \frac{e^x \, dx}{e^{2x}+1}$.

8. Evaluate $\int x \tan^{-1} x \, dx$.

9. Evaluate $\frac{\sqrt{x}}{\sqrt{x}+2}$.

10. Evaluate $\int_2^4 \frac{x}{x^2+1} dx$

11. Evaluate $\int_0^1 \frac{2x}{5x^2+1} dx$

12. Evaluate $\int_{-4}^4 |x+2| \, dx$

13. Evaluate $\int_0^1 \cot^{-1}(1-x+x^2) \, dx$

14. Evaluate $\int_0^a x^2 \sqrt{\frac{a-x}{a+x}} \, dx$

15. Show that $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x} dx = \frac{\pi}{4} - \frac{1}{2} \log 2$

EXERCISE – II

1. Evaluate $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$.

2. Evaluate $\frac{1}{\sqrt{1-2x}+\sqrt{3-2x}}$.

3. Evaluate $\sin^3 x\cos^5 x$.

4. Evaluate $\frac{x^2\left(\tan^{-1}x^3\right)}{1+x^6}$.

5. Evaluate $\int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

6. Evaluate $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$

7. Evaluate $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$

8. Evaluate $\int_2^3 \frac{dx}{(x+1)\sqrt{x^2-1}}$

9. Evaluate $\int \sin 2x \log\cos x\,dx$

10. Evaluate $\frac{\sin^6 x+\cos^6 x}{\sin^2 x\cdot\cos^2 x}$.

11. Evaluate $\frac{1}{1+3e^x+2e^{2x}}$.

12. Evaluate $\int_0^{\pi/4}\sec^4 x\,dx$.

13. Evaluate $\int_0^1 x\tan^{-1}x\,dx$

14. Evaluate $\int_0^\pi x\sin x\cos^4 x$

15. Evaluate $\int_{\pi/6}^{\pi/3}\frac{1}{1+\tan x}dx$

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\ln\left|\frac{1}{\sin x+\cos x}\right|+c$

PP2. $\tan^{-1}\left(x+\frac{1}{x}\right)+c$

PP3. $-2\ln\left|1+\frac{1}{\sqrt{x}}\right|+c$

PP4. $\ln\left(\frac{(\tan x+2)^4}{(\tan x+1)^2}\right)+c$

PP5. $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+c$

PP6. $e^x\cos x+c$

PP7. $\ln\left|(x+2)+\sqrt{x^2+4x+5}\right|+c$

PP8. $2\sqrt{x^2+x+1}-4\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$

PP9. $-\ln\left|\frac{1}{x}+\frac{1}{2}+\sqrt{\frac{1}{x^2}+\frac{1}{x}+1}\right|+c$

PP10. $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+c$

PP11. $\frac{1}{2}\ln\left|\frac{\tan x}{\tan x+2}\right|+c$

PP12. $\ln\left|\tan x+\sqrt{\tan^2 x-1}\right|+c$

PP13. 1/6

PP14. $\frac{64}{231}$

PP15. 29

PP16. $\frac{\pi}{4}$

PP17. $\frac{1}{(n+1)(n+2)}$

PP18. $\frac{\pi}{2\sqrt{2}} \ln(1+\sqrt{2})$

PP19. $\frac{\pi}{2} - 1$

ANSWERS TO EXERCISE – I

1. $2\sqrt{\tan x}$

2. $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)$

3. $\frac{1}{b^2-a^2}\log(a^2\cos^2 x+b^2\sin^2 x)$

4. $\frac{1}{\log_e 3}\sin^{-1}(3^x)$

5. $2\sin\sqrt{x}+c$

6. $\log\sec(\log x)+c$

7. $\tan^{-1}(e^x)+c$

8. $\left(\frac{x^2+1}{2}\right)\tan^{-1}x-\frac{x}{2}+c$

9. $x-4\sqrt{x}+8\log(2+\sqrt{x})+c$

10. $\frac{1}{2}\log\left(\frac{17}{5}\right)$

11. $\frac{1}{5}\log 6$

12. 20

13. $\frac{\pi}{2}-\log 2$

14. $a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$

ANSWERS TO EXERCISE – II

1. $x - \sqrt{1-x^2}\ \sin^{-1} x\ + c$

2. $\frac{1}{6}\left[(1-2x)^{3/2} - (3-2x)^{3/2}\right] + c$

3. $-\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8}\ + c$

4. $\frac{1}{6}\{\tan^{-1}(x^3)\}^2 + c$

5. $\frac{6-\pi\sqrt{3}}{12}$

6. $\frac{(\beta-\alpha)^2\pi}{8}$

7. $\frac{(\beta-\alpha)\pi}{2}$

8. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{6}}$

9. $\frac{1}{2}\cos^2 x - \cos^2 x \log\cos x + c$

10. $\tan x - \cot x - 3x + c$

11. $x + \log(1+e^x) - 2\log(1+2e^x) + c]$

12. 4/3

13. $\frac{\pi}{2} - \frac{1}{2}$

14. $\frac{\pi}{5}$

15. $\frac{\pi}{12}$