LESSON 6

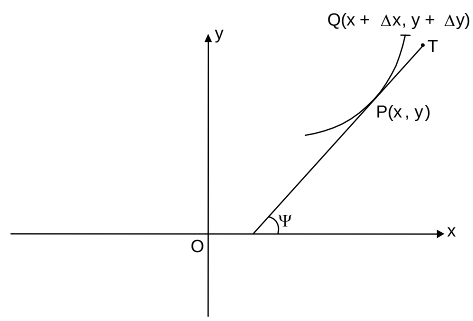
APPLICATIONS OF DERIVATIVE

1. GEOMETRICAL INTERPRETATION OF DERIVATIVE

Consider a curve $y = f(x)$ and two points $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ on it. Then $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$= \lim_{\Delta x \to 0} \left(\frac{y + \Delta y - y}{x + \Delta x - x} \right)$

$= \lim_{\Delta x \to 0}$ slope of the chord PQ



= Slope of the tangent PT at $P(x, y)$

= tan ψ, where ψ is the angle which the tangent at P makes with the positive direction of the x-axis.

***Illustration* 1**

Question: **Find the slope of tangent at the point having ordinate J3 on the curve $x^3 = 3y^2$.**

***Solution*:** Differentiating the given equation of curve w.r.t x, we get

$$3x^2 = 3.2y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2y}$$

So, we require abscissa also to obtain this value.

Substituting $y = -3$ in the equation of curve, we have

$$x^3 = 3\,(-3)^2 = 27$$

$$\Rightarrow \quad x = 3$$

$\therefore$ The point of interest is (3, –3)

Hence, slope of tangent at this point $= \left.\frac{dy}{dx}\right|_{(3,-3)} = \frac{3^2}{2(-3)} = -\frac{3}{2}$

2. TANGENT AND NORMAL

Tangent at a point is a line which touches the curve at that point and normal at a point is a line which is perpendicular to the tangent at that point.

Given the equation of a curve $y = f(x)$ and a point $A\ (x_1, y_1)$ on it, the equation of the tangent at A is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{at\,A}(x - x_1)$$

and the equation of the normal at A is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{at\,A}}(x - x_1)$$

When the curve is given in parametric form

i.e., $x = g(t)$ and $y = h(t)$

Equation of tangent at the point $t = t_1$ is

$$y - h(t_1) = \frac{h'(t_1)}{g'(t_1)}(x - g(t_1))$$

and equation of normal is $y - h(t_1) = -\frac{g'(t_1)}{h'(t_1)}(x - g(t_1))$

Illustration 2

Question: **Find the points on the curve $y = x^3 - x^2 - x + 3$ where the tangent is parallel to the x-axis.**

***Solution*:** Given curve $y = x^3 - x^2 - x + 3$

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$

Since the tangent is parallel to the x-axis, slope = tan 0° = 0

i.e., $\frac{dy}{dx} = 0$

Hence $3x^2 - 2x - 1 = 0$ or $(3x + 1)(x - 1) = 0$

$\therefore$ $x = -\frac{1}{3}$ or 1

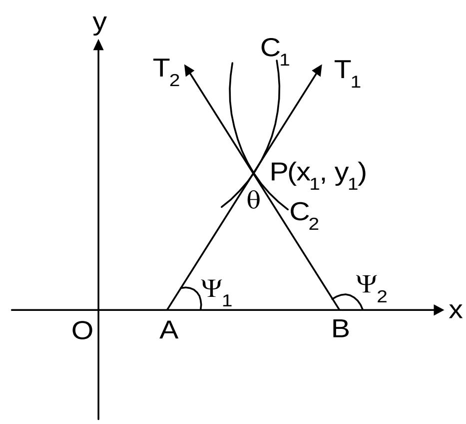
For the first point, $x = -\frac{1}{3}$ and $y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 3 = \frac{88}{27}$

For the second point, $x = 1$ and $y = 1^3 - 1^2 - 1 + 3 = 2$

Hence the points are $\left(-\frac{1}{3}, \frac{88}{27}\right)$ and (1, 2)

3. ANGLE BETWEEN TWO CURVES

Given two curves $C_1 : y = f(x)$ and $C_2 : y = g(x)$ intersecting at some point $P\ (x_1, y_1)$.



Let PT_1 be the tangent at P to curve C_1 and let PT_1 make an angle ψ_1 with OX. Let PT_2 be the tangent at P to curve C_2 and let PT_2 make an angle ψ_2 with OX. The angle between two curves is defined to be the angle between the two tangents at the point of intersection.

∴ θ, the angle between the curves

$= T_1PT_2 = \psi_2 - \psi_1$ from ΔABP

$$\tan\theta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2 \tan\psi_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If θ is the acute angle between the two curves

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$ where $m_1 = f'(x)$ at P and $m_2 = g'(x)$ at P.

Remark:

(i) curves intersect orthogonally if $m_1 m_2 = -1$;

(ii) curves touch each other if $m_1 = m_2$.

***Illustration* 3**

Question: **Find the angle of intersection of the curves $y = x^3$ and $6y = 7 - x^2$.**

Solution: The point of intersection is found by solving the equations simultaneously

$$y = x^3 \text{ and } y = \frac{7}{6} - \frac{x^2}{6}$$

∴ $6x^3 = 7 - x^2$ or $6x^3 + x^2 - 7 = 0$

$(x - 1)(6x^2 + 7x + 7) = 0$

This gives $x = 1$ only, the other factor gives complex roots when $x = 1$, $y = 1$, on substitution in $y = x^3$

Now, from C_1: $\frac{dy}{dx} = 3x^2$ = 3 at x = 1

From C_2: $\frac{dy}{dx} = \frac{-2x}{6} = \frac{-1}{3}$ at x = 1

Since the product of the slopes = –1, the curves intersect at right angles.

4. DERIVATIVE AS RATE MEASURE

This article shows, how derivative is useful in determination of rates of change related to physical situations.

***Illustration* 4**

Question: **A spherical balloon is pumped at the rate of 10 cubic inches per minute, find the rate of increase of its radius when its radius is 15 inches.**

***Solution*:** Let y be the volume and x the radius of the balloon at any time t.

Given, $\frac{dy}{dt} = 10$ cubic inches per minute.

To find $\frac{dx}{dt}$ when x = 15 inches.

Since the balloon is spherical $\therefore\ y = \frac{4}{3}\pi x^3$ …(i)

$$\frac{dy}{dt} = \frac{4}{3}\pi.3x^2\frac{dx}{dt} = 4\pi x^2\frac{dx}{dt} \quad \text{…(ii)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{dy/dt}{4\pi x^2} = \frac{10}{4\pi x^2}$$

$\therefore$ when x = 15 inches, $\frac{dx}{dt} = \frac{10}{4\pi.15^2}$

$= \frac{1}{90\pi}$ inch per minute.

Hence rate of increase of its radius when radius is 15 inches is $\frac{1}{90\pi}$ inch/minute.

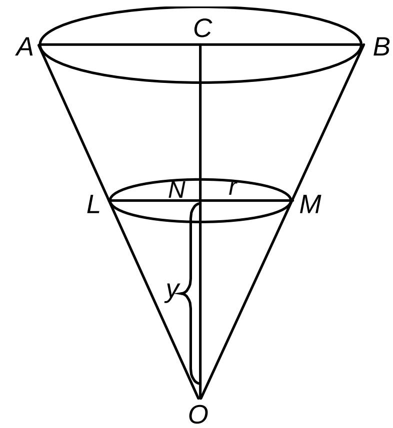
***Illustration* 5**

Question: **A right circular cone is 10 inches in diameter and 10 inches deep. Water is poured into it at the rate of 4 cubic inches per minute. At what rate is the water level rising at the instant when the depth is 6 inches.**

Solution: Let OAB be the cone and LM be the level of water at any time t.

Let $ON = y$ and volume $OLM = x$, $MN = r$, given $AB = 10$ inches $OC = 10$ inches and $\frac{dx}{dt} = 4$ cubic inches/minute.

We have to find $\frac{dy}{dt}$ when $y = 6$ inches.



Now, $x = \frac{1}{3}\pi r^2.y$ …(i)

$$\left[\because \text{ volume of a cone } = \frac{1}{3}\pi r^2 h\right]$$

Now from similar ΔONM and ΔOCB

$$\frac{MN}{BC} = \frac{ON}{OC} \quad \text{or,} \quad \frac{r}{5} = \frac{y}{10} \quad \text{or,} \quad r = \frac{y}{2}$$

Putting $r = \frac{y}{2}$ in (i), we get

$$x = \frac{1}{3}\pi\frac{y^2}{4}.y = \frac{\pi}{12}y^3$$

differentiating w.r.t., t we get

$$\frac{dx}{dt} = \frac{\pi}{12}.3y^2\frac{dy}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dx}{dt}$$

$\therefore$ when $y = 6$ inches

$$\frac{dy}{dt} = \frac{4}{\pi.6^2}4 = \frac{4}{9\pi} \text{ inches/minute}$$

Hence the water level is rising at the rate $\frac{4}{9\pi}$ inches/minute when the depth of water is 6 inches.

5. MONOTONICITY OF FUNCTIONS

In this section we shall study the behaviour of functions. Basically we have four kinds of behaviours shown in functions in intervals of their domains.

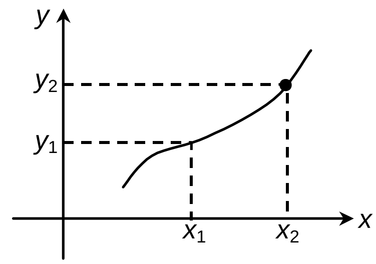
5.1 INCREASING BEHAVIOUR

If in an interval ***I***, for any two points (x_1, y_1) and (x_2, y_2) we have

$$x_2 > x_1 \Leftrightarrow y_2 > y_1$$

the function is said to be monotonically increasing or simply increasing in ***I***.

If the function is differentiable in the interval of interest (which is normally true for most of the functions), it can be inferred that $\frac{dy}{dx} > 0$ for all points in that interval.



Caution: This condition normally suffices to find intervals of increase but sometimes the derivative may be zero also at specific points in the interval and still the function may be increasing.

e.g., $y = x^3$

$\Rightarrow \quad \frac{dy}{dx} = 3x^2$

Now $\frac{dy}{dx} > 0$ for all real x except $x = 0$.

$\therefore$ Here $\frac{dy}{dx} \geq 0$ for the entire domain but still the function is increasing. For any two points such that $x_2 > x_1$, we have $y_2 > y_1$ certainly.

5.2 DECREASING BEHAVIOUR

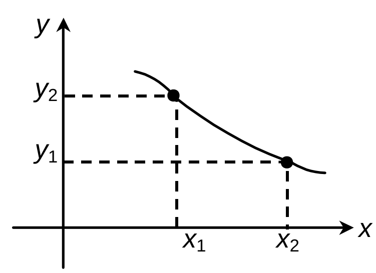
If in an interval ***I***, for any two points (x_1, y_1) and (x_2, y_2) we have

$$x_2 > x_1 \quad \Leftrightarrow \quad y_2 < y_1$$

the functions is said to be monotonically decreasing or simply decreasing in ***I***.

Again for differentiable function, here $\frac{dy}{dx} < 0$ for all points in the interval.

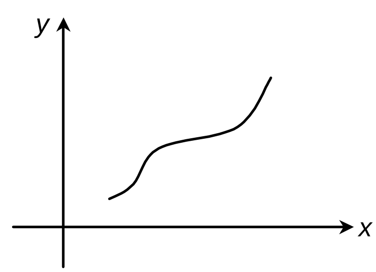
The same '**Caution**' holds here also.



5.3 NON-DECREASING BEHAVIOUR

Here $x_2 > x_1 \Leftrightarrow y_2 \geq y_1$, for all points in that interval.

$\Rightarrow \quad \frac{dy}{dx} \geq 0$, where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.



5.4 NON-INCREASING BEHAVIOUR

In this case,

$x_2 > x_1 \quad \Leftrightarrow \quad y_2 \le y_1$

for all points in that interval.

$\Rightarrow \quad \frac{dy}{dx} \le 0$, where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.

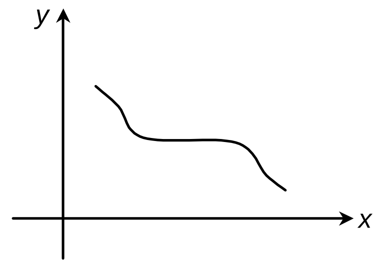


Illustration 6

Question: **Find the values of x for which the function $f(x) = 2x^3 - 21x^2 + 72x + 30$ is (i) increasing, (ii) decreasing.**

Solution: $f'(x) = 6x^2 - 42x + 72 = 6(x^2 - 7x + 12) = 6(x-3)(x-4)$

(i) $f(x)$ is increasing if $f'(x) > 0$

if $6(x-3)(x-4) > 0$

if either $x > 4$ or $x < 3$

if $x \in (-\infty, 3) \cup (4, \infty)$

(ii) $f(x)$ is decreasing if $f'(x) < 0$

if $6(x-3)(x-4) < 0$

if $x \in (3, 4)$

6. MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE

Definition: Let $f(x)$ be defined on an interval A. Let a and $b \in A$. Then

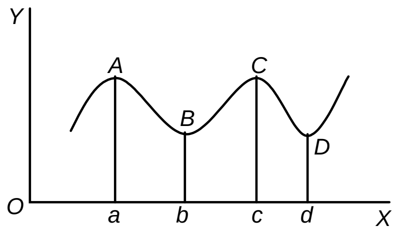
(i) $f(x)$ is said to have a maximum value at $x = a$ if $f(a) > f(a+h)$ and $f(a) > f(a-h)$ for all sufficiently small positive values of h. The point a is called the point at which the function is a maximum and $f(a)$ is the corresponding maximum value of the function.

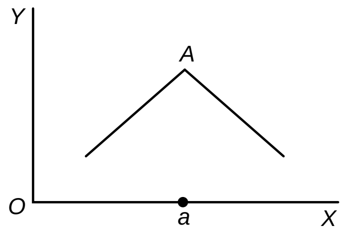
(ii) $f(x)$ is said to have a minimum value at $x = b$ if $f(b) < f(b+h)$ and $f(b) < f(b-h)$ for all sufficiently small positive values of h. The point b is called the point at which the function is a minimum.

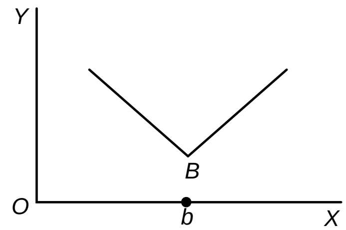
Maximum and minimum values of $f(x)$ as defined above are not necessarily the greatest and least values of $f(x)$. They are maximum and minimum in the immediate neighbourhood of $x = a$ and $x = b$. Hence these are also referred as local maximum or local minimum.

The points of maximum or minimum of a function are also called the points of extremum.

A necessary condition for the existence of an extremum (maximum or minimum) for a function $f(x)$ is either $f'(x) = 0$ or $f'(x)$ does not exist.







In the first figure, at the maximum and minimum points on the graph, the tangent is parallel to the x-axis and hence $f'(x) = 0$. In the second figure, the function is increasing as x approaches a from the left and is decreasing as x increases beyond a. The graph is not smooth and hence has no tangent at $x = a$.

6.1 CRITERIA FOR MAXIMA AND MINIMA

- If $f(x)$ has a maximum value at $x = a$ and $f'(a)$ exists, then $f'(a)$ must be zero. Similarly, if $f(x)$ has a minimum value at $x = b$ and $f'(b)$ exists, then $f'(b)$ must be zero.
- If c be a point in the domain of $f(x)$ such that

  $f'(c) = 0$ and $f''(c) \neq 0$, then

  - $f(c)$ is a maximum if $f''(c) < 0$
  - $f(c)$ is a minimum if $f''(c) > 0$

6.2 ALTERNATIVE CRITERIOA FOR MAXIMA AND MINIMA

- If $f'(a) = 0$ and $f'(x)$ changes sign from plus to minus as x passes through the point 'a' from left to right, then $f(x)$ is maximum at $x = a$.
- If $f'(b) = 0$ and $f'(x)$ changes sign from minus to plus as x passes through the point 'b' from left to right, then $f(x)$ is minimum at $x = b$.

If the derivative does not change sign in moving from left to right through the point a, then $f(x)$ has neither maximum nor minimum at $x = a$.

6.3 n^{th} DERIVATIVE TEST

(It can be applied to $x = c$ only if $f'(c) = 0$ and $f''(c) = 0$).

By differentiation, find nth derivative of $f(x)$ denoted by $f^n(x)$, $n \in \boldsymbol{N}$

Step-by-step, find the earliest non-zero $f^n(c)$, $n = 3, 4, 5, 6, 7, \ldots\ldots\ldots$

In this process,

(i) if n is odd $\Rightarrow$ $x = c$ is neither local maximum nor local minimum point.

(ii) if n is even, and if

$$f^n(c) = \begin{cases} +ve \quad , \Rightarrow x = c \text{ is local minimum point} \\ -ve \quad , \Rightarrow x = c \text{ is local maximum point.} \end{cases}$$

6.4 ABSOLUTE MAXIMUM / MINIMUM POINTS

- To find absolute maximum / minimum values of $f(x)$ in open interval (a, b), we proceed as follows:
  - Find all extremum points of $f(x)$ by using critical points. Let these extremum points be c_1, c_2, c_3 ……….
  - Compare the lengths of ordinates $f(c_1)$, $f(c_2)$, $f(c_3)$, ………….
  - The greatest value of these ordinates is called absolute maximum value of $f(x)$.
  - The least value of these ordinates is called absolute minimum value of $f(x)$.

  Absolute maximum value/absolute minimum value can occur at more than one extremum point. Absolute maximum/minimum value is also called Global maximum/minimum value of $f(x)$.
- To find absolute maximum/minimum value in closed interval $[a, b]$, include values of ordinates at the end points viz. $f(a)$ and $f(b)$, in the above procedure of comparison of lengths of the ordinates at the extremum points.

Illustration 7

Question: **Find the local maximum and minimum value of $f(x) = 2x^3 - 15x^2 + 36x + 11$.**

Solution: Let $y = 2x^3 - 15x^2 + 36x + 11$

$$\frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$$

$$\frac{d^2y}{dx^2} = 12x - 30$$

For extremum, $\frac{dy}{dx} = 0$

i.e. $x^2 - 5x + 6 = 0$

i.e. $(x - 2)(x - 3) = 0$

i.e. $x = 2$ or $x = 3$

Now $\frac{d^2y}{dx^2}$ (at $x = 2$) $= 12(2) - 30 = -6$, negative

$\therefore$ y or $f(x)$ is a maximum when $x = 2$ and the maximum value of $f(x) = f(2)$

$= 2(2^3) - 15(2^2) + 36(2) + 11 = 39$

$\frac{d^2y}{dx^2}$ (at $x = 3$) $= 12 \times 3 - 30 = +6$, positive

$\therefore$ y or $f(x)$ is a minimum when $x = 3$ and

the minimum value of $f(x) = f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 11 = 38$

***Illustration* 8**

Question: **Find local maximum / minimum points of $f(x) = (x \text{ J } 2)^3 (x \text{ J } 3)$.**

Solution:

$f'(x) = (x - 2)^2 (4x - 11) = 0 \Rightarrow x = 2, 2, 11/4$ are critical points.

$f''(x) = 2(x - 2)(4x - 11) + 4(x - 2)^2$.

$f''(11/4) > 0 \Rightarrow x = 11/4$ is local minimum point.

$f''(2) = 0 \Rightarrow$ Second derivative test fails.

$f'''(x) = 2(4x - 11) + 16(x - 2)$

$f''(2) = -6 \neq 0$.

At earliest non-zero derivative at $x = 2$ is of odd order, $x = 2$ is neither local maximum nor local minimum point.

Hence $x = 11/4$ is the only local minimum point of $f(x)$.

PRACTICE PROBLEMS

PP1. Find the slopes of the curve $y = (x + 2)(x - 3)$ at the points where it meets the x-axis.

PP2. Find the points on the curve $y = x^3 - 2x^2 + x - 2$ when the gradient is zero.

PP3. Find the angle of intersection of the curves $y = x^2$ and $y = x^3$.

PP4. Prove that the curves $x^2 - y^2 = 16$ and $xy = 25$ cut each other at right angles.

PP5. If the radius of a circle is increasing at a constant rate of 2 ft./sec., find the rate of increase of its area when the radius is 20 ft.

PP6. Water is poured at the rate of 1 cubic ft. per minute into a cylindrical tub. If the tub has a circular base of radius a ft., find the rate at which water is rising in the tub.

PP7. Separate the intervals in which the function $f(x) = x - e^x$ is increasing and decreasing.

PP8. Prove that $x > \sin x$ for all $x \in (0, \infty)$.

PP9. Prove that the function $f(x) = \sin x + \sqrt{3}\cos x$ has a maximum value at $x = \frac{\pi}{6}$.

PP10. Find the global maximum value of $f(x) = x^2 - 4x + 20$ in the interval $[0, 5]$.

PP11. Find the minimum value of x^x.

PP12. Find the angle of intersection of the curves $y = x^2$ and $6y = 7 - x^2$.

PP13. Find the interval in which the function $f(x) = 2\ln(x - 2) - (x^2 - 4x - 1)$ increases.

PP14. Prove that among all rectangles with given perimeter, the square has maximum area.

PP15. Find the range of the function $f(x) = x^3 - 3x^2 + 6x - 2$ where $x \in [-1, 1]$.

SOLVED SUBJECTIVE EXAMPLES

Example 1:

Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

Solution:

Let $y = \left(\frac{1}{x}\right)^x$

$\log y = x \log \left(\frac{1}{x}\right) = -x \log x$ …(i)

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -\left[x \cdot \frac{1}{x} + \log x\right] = -(1+\log x)$ …(ii)

Differentiating again

$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{1}{x}$ …(iii)

From (ii) $\frac{dy}{dx} = -y\,(1+\log x) = -\left(\frac{1}{x}\right)^x (1+\log x)$

For maximum or minimum values of y, $\frac{dy}{dx} = 0$

$\therefore \quad \left(\frac{1}{x}\right)^x (1+\log x) = 0$; but $\left(\frac{1}{x}\right)^x \neq 0$ for any x

$\therefore \quad 1 + \log x = 0$; $\log_e x = -1$; $x = e^{-1} = \frac{1}{e}$ …(iv)

when $x = \frac{1}{e}$ equation (iii) gives

$\frac{1}{y}\frac{d^2y}{dx^2} - 0 = -e \quad \therefore \; \frac{d^2y}{dx^2} = -e\,(e)^{\frac{1}{e}}$ = negative

Hence y is maximum when $x = \frac{1}{e}$

and the maximum value of $y = e^{\frac{1}{e}}$

Example 2:

Find both the maximum and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval [1, 4].

Solution:

Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 1$, then

$f'(x) = 12x^3 - 24x^2 + 24x - 48$ and $f''(x) = 36x^2 - 48x + 24$

Now $f'(x) = 0 \Rightarrow 12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow \quad x^3 - 2x^2 + +2x - 4 = 0 \Rightarrow x^2(x-2) + 2(x-2) = 0$

$\Rightarrow \quad (x-2)(x^2+2) = 0 \Rightarrow x = 2 \qquad [\because \; x^2 + 2 \neq 0]$

For $x = 2$, $f''(x) = 36(2)^2 - 48(2) + 24 = 72 > 0$

So, $x = 2$ is a point of local minimum.

Now $f(2) = -59$, $f(1) = -40$ and $f(4) = 257$

So, the minimum and maximum value of $f(x)$ on $[1, 4]$ are -59 and 257 respectively.

Example 3:

Find the points of local extrema of the function *f*(*x*) = (2*x* J 1)$^{2/5}$ (*x* + 2).

Solution:

$f'(x) = (2x-1)^{2/5}(1) + (x+2)\,\frac{2}{5}\,(2x-1)^{-3/5}(2)$

$= (2x-1)^{2/5} + \frac{4(x+2)}{5(2x-1)^{3/5}}$

$= \frac{(2x-1) + 4x + 8}{5(2x-1)^{3/5}}$

$= \frac{6x+7}{5(2x-1)^{3/5}}$

For critical points $f'(x) = 0$ or undefined

$\therefore \quad x = -\frac{7}{6}, \frac{1}{2}$

Around $x = -\frac{7}{6}$, the sign of $f'(x)$ changes from positive to negative.

$\therefore$ We have a local maxima at $x = -\frac{7}{6}$.

Around $x = 1/2$, the sign of $f'(x)$ changes from negative to positive.

$\therefore$ We have a local minima at $x = 1/2$.

Example 4:

Find the right circular cone of maximum volume that can be inscribed in a sphere of radius *R*.

Solution:

Let *ABC* be the cone with radius *R*

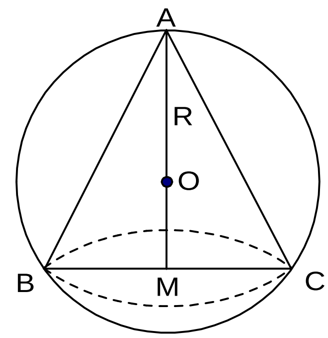
$BM = MC = x$ and height $AM = y$

In ΔOMB, $BM^2 + OM^2 = OB^2$

$x^2 + (y - R)^2 = R^2$

$\therefore \quad x^2 = 2Ry - y^2$

Volume (V) of the cone



$$= \frac{1}{3}\pi x^2 y = \frac{\pi}{3} y\,(2Ry - y^2)$$

$$= \frac{\pi}{3}(2Ry^2 - y^3)$$

$$\frac{dV}{dy} = \frac{\pi}{3}(4Ry - 3y^2) = \frac{\pi}{3} y\,(4R - 3y)$$

$\frac{dV}{dy} = 0 \quad \Rightarrow y = \frac{4R}{3}$, ($y$ = 0 is meaningless in this context)

$\frac{d^2V}{dy^2} = \frac{\pi}{3}(4R - 6y)$ and $\frac{d^2V}{dy^2}$ when $y = \frac{4R}{3}$ is $\frac{\pi}{3}(4R - 8R)$ is negative

$\therefore$ V is maximum when $y = \frac{4R}{3}$

Cone has maximum volume when height is $\frac{4R}{3}$ and radius is $\frac{2\sqrt{2}R}{3}$.

Example* 5*:

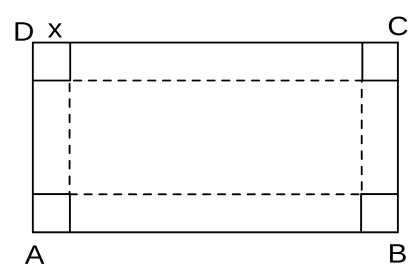
A rectangular sheet of metal has four equal square portions removed from the four corners and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is maximum the depth will be $\frac{1}{6}\left[(a + b) - \sqrt{a^2 - ab + b^2}\right]$ where a and b, ($a > b$) are the sides of the original rectangle.

Solution:

Let $ABCD$ be the given rectangular sheet of metal with $AB = a$, $BC = b$ and x be the side of the four squares cut off..

Volume (V) of the box = $(a - 2x)(b - 2x)\,x$

$V = abx - 2(a + b)x^2 + 4x^3$



$$\frac{dV}{dx} = ab - 4x(a + b) + 12x^2$$

$$\frac{d^2V}{dx^2} = 24x - 4(a + b)$$

$\frac{dV}{dx} = 0$ when $12x^2 - 4(a + b)x + ab = 0$

or when $x = \dfrac{4(a + b) \pm \sqrt{16(a + b)^2 - 48ab}}{24}$

$$= \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$$

The plus sign gives a value of x greater than $\frac{b}{2}$ and hence not admissible.

$\frac{d^2V}{dx^2}$ when $x = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}$ is negative.

$\therefore$ V is maximum when $x = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}$

***Example* 6:**

Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is decreasing or increasing.

Solution:

We have, $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$\Rightarrow \quad f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6)$

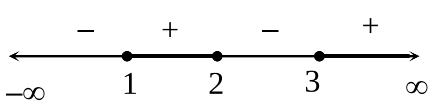
$\Rightarrow \quad f'(x) = 4(x-1)(x^2 - 5x + 6)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0 \Rightarrow 4(x-1)(x^2 - 5x + 6) > 0$

$\Rightarrow \quad (x-1)(x^2 - 5x + 6) > 0$ $\quad\quad$ $[\because \; 4 > 0]$

$\Rightarrow \quad (x-1)(x-2)(x-3) > 0 \Rightarrow x \in (1, 2) \cup (3, \infty)$



So, $f(x)$ is increasing on $(1, 2) \cup (3, \infty)$

And $f(x)$ to be decreasing $(2, 3) \cup (-\infty, 1)$

***Example* 7:**

Show that the curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$ cut each other orthogonally where a and b are constants.

Solution:

C_1: $\quad x^3 - 3xy^2 = a \Rightarrow 3x^2 - 3y^2 - 6xy \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ $\quad\quad$ …(i)

C_2: $\quad 3x^2y - y^3 = b \Rightarrow 6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{2xy}{y^2 - x^2}$ …(ii)

Hence the product of the slopes of the tangents to the two curves at the point of intersection

$$= \frac{x^2 - y^2}{2xy} \times -\frac{2xy}{x^2 - y^2} = -1$$

Hence the two curves intersect at right angles (or orthogonally).

***Example* 8:**

Prove that the sum of the intercepts of the tangent to $\sqrt{\boldsymbol{x}}\,\mathrm{H}\sqrt{\boldsymbol{y}}\,\mathrm{Z}\sqrt{\boldsymbol{a}}$ upon the coordinate axes is constant.

Solution:

Curve: $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating w.r.t. x, $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \qquad \therefore \left(\frac{dy}{dx}\right) \text{ at } P(x_1, y_1) = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

Equation of the tangent at P (x_1, y_1) is $\quad y - y_1 = \frac{-\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$

i.e. $\sqrt{y_1}\,x + \sqrt{x_1}\,y = \sqrt{x_1 y_1}\left(\sqrt{x_1} + \sqrt{y_1}\right) = \sqrt{x_1 y_1 a}$ (since (x_1, y_1) is a point on the curve)

or $\frac{x}{\sqrt{ax_1}} + \frac{y}{\sqrt{ay_1}} = 1$, in the intercept form is the equation of the tangent.

Sum of the intercepts = $\sqrt{ax_1} + \sqrt{ay_1} = \sqrt{a}\left(\sqrt{x_1} + \sqrt{y_1}\right) = \sqrt{a}\,\sqrt{a} = a$, a constant.

***Example* 9:**

Find the equation of tangent line to $\boldsymbol{y}\,\mathrm{Z}\,\mathbf{2}\boldsymbol{x}^{\mathbf{2}}\,\mathrm{H}\,\mathbf{7}$ which is parallel to the line $\mathbf{4}\boldsymbol{x}\,\mathrm{J}\,\boldsymbol{y}\,\mathrm{H}\,\mathbf{3}\,\mathrm{Z}\,\mathbf{0}$.

Solution:

Let the point of contact of the required tangent line by $(x_1,\ y_1)$.

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 4x \;\Rightarrow\; \left(\frac{dy}{dx}\right)_{(x_1,\ y_1)} = 4x_1$$

Since the line $4x - y + 3 = 0$ is parallel to the tangent at $(x_1,\ y_1)$

$\therefore$ Slope of the tangent at (x_1, y_1) = slope of the line $4x - y + 3 = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4}{-1} \Rightarrow 4x_1 = 4 \Rightarrow x_1 = 1$$

Now (x_1, y_1) lies on $y = 2x^2 + 7$

$$\therefore y_1 = 2x_1^2 + 7 \Rightarrow y_1 = 2 + 7 = 9 \quad [\because x = 1]$$

So, the coordinates of the point of contact are (1, 9). Hence the required equation of the tangent line is $y - 9 = 4(x - 1) \Rightarrow 4x - y + 5 = 0$

***Example* 10:**

A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semicircle. The semi circular portion is fitted with colour glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits maximum light.

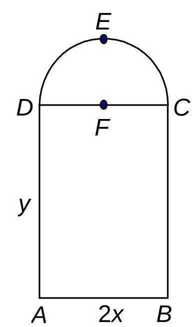
Solution:

Let $2x$, y be the dimensions of the rectangular portion.

Then the radius of the semi circular portion is x.

Perimeter = $2x + 2x + 2y + \pi x = k$

$$\therefore \quad y = \frac{k - x(4+\pi)}{2} \quad \ldots\text{(i)}$$



Let 'a' units of light per square unit area of colour glass be transmitted. Then '$3a$' units will be transmitted by clear glass.

Amount of light (L) transmitted = $\left(\frac{\pi x^2}{2}\right)a + 2xy\,(3a)$

$$= \frac{a\pi}{2}x^2 + 6ax\left\{\frac{k - x(4+\pi)}{2}\right\} \quad \ldots\text{(ii)}$$

$$\frac{dL}{dx} = a\pi x + 3ak - 6ax(4+\pi); \quad \frac{d^2L}{dx^2} = a\pi - 6a(4+\pi)$$

$$\frac{dL}{dx} = 0 \text{ when } x = \frac{3ak}{24a + 5a\pi} = \frac{3k}{24 + 5\pi}; \qquad \frac{d^2L}{dx^2} = -a(12 + 5\pi) = \text{negative}$$

$\therefore L$ is maximum when $x = \dfrac{3k}{24 + 5\pi}$

$$\text{Ratio of the sides of the rectangle} = \frac{2x}{y} = \frac{\dfrac{6k}{24+5\pi}}{\dfrac{k - \dfrac{3k(4+\pi)}{24+5\pi}}{2}} = \frac{12k}{12k + 2k\pi} = \frac{6}{6+\pi}$$

EXERCISE – I

1. The radius of a spherical air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when its radius is 1 cm?

2. Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.

3. Find the equation of the tangent to the curve $x^2 + 3y = 3$ which is parallel to the line $y - 4x + 5 = 0$.

4. For the curve $y = 4x^3 - 2x^5$, find all points at which the tangent passes through the origin.

5. Find the point at which the line $y = 3x + 8$ is parallel to the tangent to the curve $y = 2x^2 - x + 1$.

6. Find the domain of $f(x)$ in which $f(x) = \tan^{-1} x - x$ is monotonically decreasing.

7. Find the maximum value of $\left(\frac{1}{x}\right)^{x^2}$.

8. Find the intervals in which the function $f(x) = (x+1)^3 (x-3)^3$ is increasing or decreasing.

9. Find the intervals in which the function $f(x) = \sin x + \cos x$ in $[0, 2\pi]$ is increasing or decreasing.

10. Show that the function $f(x) = 3x^5 + 40x^3 + 240x$ is always increasing on R.

11. Find the absolute maximum of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[0, 4]$

12. A right circular cylinder is inscribed in a given cone. Show that the curved surface area of cylinder is maximum when diameter of cylinder is equal to radius of base of cone.

13. Show that volume of greatest cylinder which can be inscribed in a cone of height h and semi vertical angle α is $4/27\,\pi h^3 \tan^2\alpha$.

14. Using differentials, find the approximate value of

(i) $\sqrt[3]{29}$ (ii) $\sqrt{0.26}$. (iii) $\sqrt{0.37}$ (iv) $\sqrt{0.82}$

15. If $y = x^4 - 10$ and if x changes from 2 to 1.97, what is the approximate change in y?

EXERCISE – II

1. Find the equation of the tangent to the curve $y = -5x^2 + 6x + 7$ at the point $\left(\frac{1}{2}, \frac{35}{4}\right)$.

2. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.

3. Find the abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts off equal intercepts from the coordinate axes.

4. Prove that $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at (1, 2).

5. Show that $f(x) = x^3 - 3x^2 - 9x + 20$ is positive for all $x > 4$.

6. Show that $f(x) = x^9 + 4x^7 + 11$ is an increasing function for all $x > 0$.

7. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$, $x > 0$ is increasing in $\left(0, \frac{\pi}{4}\right)$.

8. Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$ where the tangent of the curve has the greatest slope.

9. Find the maximum and minimum values of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3].

10. If $f(x) = \begin{cases} |x-1| + \lambda & , \quad x \le 1 \\ 2x + 3 & , \quad x > 1 \end{cases}$ has a local minimum at $x = 1$, then find the maximum value of λ.

11. Find a, b, c so that the two curves $y = x^2 + ax + b$ and $y = cx - x^2$ may touch each other at (1, 0).

12. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at P (–2, 0) and cuts the y-axis at a point Q where its gradient is 3. Find a, b, c.

13. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the function. Also find the points of inflection if $f(x) = x^4 - 62x^2 + 120x + 9$.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. A 12 cm long wire is bent to form a triangle with one of the angle is 60°. Find the sides of the triangle when the area is maximum.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. –5, 5

PP2. (1, –2) and $\left(\frac{1}{3}, -\frac{50}{27}\right)$

PP3. $\tan^{-1}\frac{1}{7}$, 0

PP5. 320π sq. ft./sec.

PP6. $\frac{1}{\pi a^2}$ ft./min.

PP7. Increasing in (–∞, 0) ; decreasing in (0, ∞).

PP10. 25

PP11. $\left(\frac{1}{e}\right)^{1/e}$

PP12. $\tan^{-1}7$

PP13. (2, 3)

PP15. [–12, 2]

EXERCISE – I

1. 2π cm^3/sec.

3. $y - 4x = 13$

4. (0, 0), (1, 2), (–1, –2)

5. (1, 2)

6. $R - \{0\}$

7. $e^{\frac{1}{2e}}$

8. increasing: $x > 1$
decreasing: $x < 1$

9. increasing: $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$

decreasing: $\frac{\pi}{4} < x < \frac{5\pi}{4}$

11. 33

14. (i) 3.074
(ii) 0.51
(iii) 0.608
(iv) 0.905

15. –0.96, y changes from 6 to 5.04

EXERCISE – II

1. $4y - 4x - 33 = 0$

2. 4π

3. $x = \frac{4a}{9}$

8. (0, 0)

9. Maximum value = 89

Minimum value = 75

10. $\lambda = 4$

11. $a = -3$, $b = 2$, $c = 1$

12. $a = -\frac{1}{2}, b = -\frac{3}{4}, c = 3$

13. Local maximum at $x = 1$, local maximum value of 68

Local minimum at $x = 5, -6$, local minimum value are –1647 and –316

14. $x = 15, y = 45$

15. Triangle is equilateral of side 4.