‘

LESSON 5

CONTINUITY AND DIFFERENTIABILITY

1. CONTINUITY

1.1 CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

i.e. L.H.L = R.H.L. = value of the function at ‘a’ i.e. $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is not continuous a $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

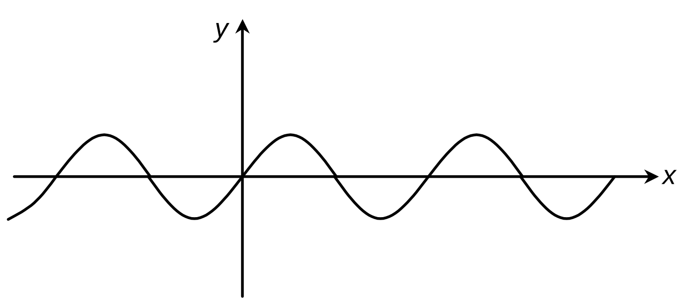
Geometrical meaning of continuity

Function $f(x)$ will be continuous at $x = c$ if there is no break in the graph of function $f(x)$ at the point $(c, f(c))$.

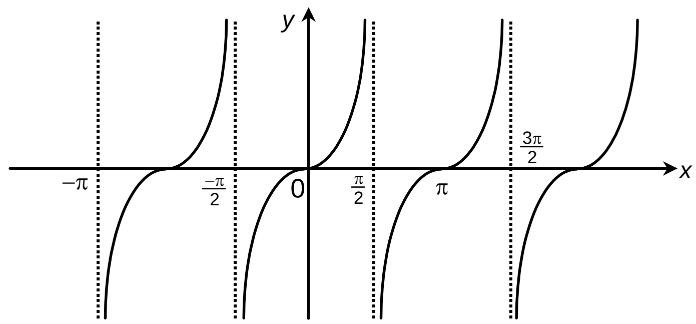
In an interval, function is said to be continuous if there is no break in graph of function in the entire interval.

For example:

- $f(x) = \sin x$ is continuous in its entire domain.



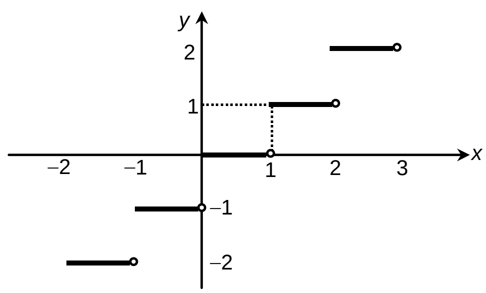
- $f(x) = \tan x$ is discontinuous at $x = (2n+1)\frac{\pi}{2}$ where $n \in \boldsymbol{I}$.



$f(x)$ will be discontinuous at $x = a$, in any of the following cases :

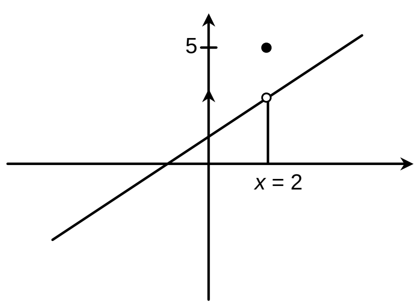
(i) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but are not equal.

For example $y = [x]$ at $x \in \boldsymbol{I}$.



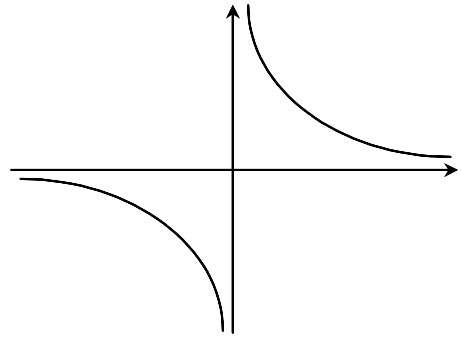
(ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but not equal to $f(a)$.

For example $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases}$ at $x = 2$.



(iii) $f(a)$ is not defined.

For example $y = \dfrac{1}{x}$ at $x = 0$



(iv) At least one of the limits does not exist.

For example $y = \sin\left(\frac{1}{x}\right)$ at $x = 0$

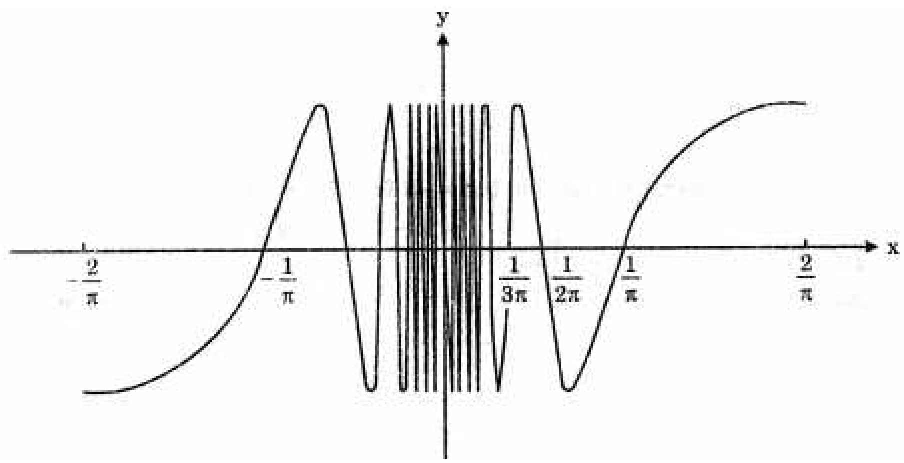


Illustration 1

Question: **$f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$ for all x in $\left(0, \frac{\pi}{2}\right)$ except at $x = \frac{\pi}{4}$. Define $f\left(\frac{\pi}{4}\right)$ so that $f(x)$ may be continuous at $x = \frac{\pi}{4}$.**

***Solution*:** $f(x)$ will be continuous at $x = \frac{\pi}{4}$ *if* $\lim_{x \to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore f\left(\frac{\pi}{4}\right)$ should be $= \lim_{x \to \pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$

$$= \lim_{x \to \pi/4} \frac{\left(\sqrt{2}\cos x - 1\right)\sin x}{\cos x - \sin x}$$

$$= \lim_{x \to \pi/4} \frac{\left(\sqrt{2}\cos x - 1\right)\left(\sqrt{2}\cos x + 1\right)}{\left(\sqrt{2}\cos x + 1\right)\left(\cos x - \sin x\right)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} . \sin x$$

$$= \lim_{x \to \pi/4} \left(\frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \right) \frac{(\cos x + \sin x)\sin x}{\left(\sqrt{2}\cos x + 1\right)}$$

$$= \lim_{x \to \pi/4} \frac{\sin x (\cos x + \sin x)}{\sqrt{2}\cos x + 1}$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

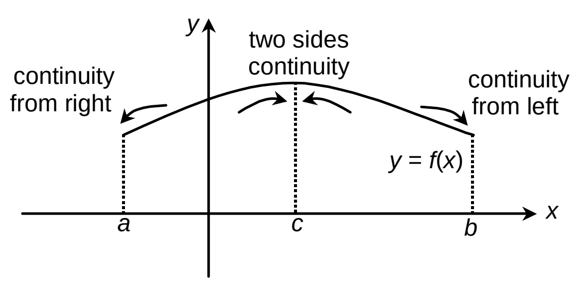
Continuity in an open interval

A function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each point of (a, b).

Continuity in a closed interval

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ if it is

- continuous at each point (a, b)



Continuity at points *a*, *b* and *c*

- $f(x)$ is continuous from right at $x = a$

  i.e. $\lim_{x \to a^+} f(x) = f(a)$

- $f(x)$ is continuous from left at $x = b$

  i.e. $\lim_{x \to b^-} f(x) = f(b)$

Illustration 2

Question: **If** $f(x) = \begin{cases} 2 + x, & x \in [0, 1] \\ 2 - x, & x \in (1, 2] \\ 4 - x, & x \in (2, 3) \end{cases}$ **examine continuity of** $f(x)$.

***Solution*:** At $x = 1$

LHL $f(1-h) = \lim_{h \to 0} 2 + (1-h) = 3$

RHL $f(1+h) = \lim_{h \to 0} 2-(1+h) = 1$

At $x = 2$

LHL $f(2-h) = \lim_{h \to 0} 2-(2-h) = 0$

RHL $f(2+h) = \lim_{h \to 0} 4-(2+h) = 2$

$f(x)$ is discontinuous at $x = 1$ and 2.

Properties of continuous functions

Let $f(x)$ and $g(x)$ are continuous functions at $x = a$. Then

(i) $c\,f(x)$ is continuous at $x = a$ where c is any constant

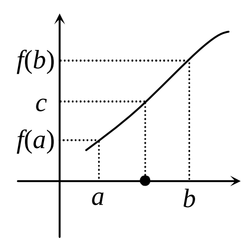
(ii) $f(x) \pm g(x)$ is continuous at $x = a$

(iii) $f(x) \cdot g(x)$ is continuous at $x = a$

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$

Intermediate value theorem

If c is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = c$ in the open interval (a, b), if $y = f(x)$ is continuous in the interval.



Types of discontinuities

Basically there are two types of discontinuity:

(i) Removable discontinuity

If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, then $f(x)$ has a removable discontinuity at $x = a$ and it can be removed by redefining $f(x)$ for $x = a$.

(ii) Non-removable discontinuity

If $\lim_{x \to a} f(x)$ does not exist, then we can not remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.

***Illustration* 3**

Question: **Redefine the function** $f(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **in such a way that it could become continuous for all** x**.**

***Solution*:** $\lim_{x\to 0+h} f(x) = 0$

$\lim_{x\to 0-h} f(x) = 0$

$f(0) = 1$

Here $f(x)$ has a removable discontinuity at $x = 0$

To remove this we defined $f(x)$ as follows:

$$f(x) = \begin{cases} x & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Now $f(x)$ is continuous at every where.

2. DIFFERENTIABILITY

Let $y = f(x)$ be a given function. If at some point, abscissa is x_1 and at other point abscissa is x_2, then it is quite natural then ordinate can be represented by y_1 and y_2 respectively at those points.

$\Delta y = y_2 - y_1$ Δy, represents change in 'y'.

$\Delta x = x_2 - x_1$ Δx, represents change in 'x'.

then $\Delta y = f(x_1 + \Delta x) - f(x_1)$

clearly increment can be positive, negative or may even be zero.

Differential coefficient of $y = f(x)$, with respect to x is defined as the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx tends to zero.

It is usually denoted by $\frac{dy}{dx}$ or $f'(x)$ symbolically.

The derivative of the function with respect to x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. i.e. $\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}$

The function is said to be differentiable at $x = a$ if

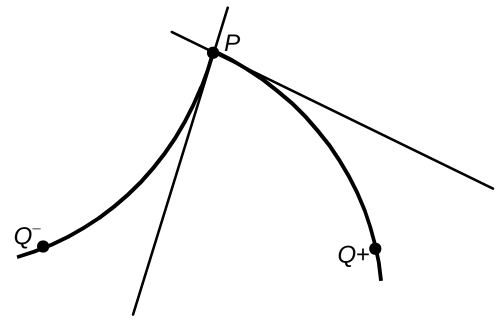
Right hand derivative (RHD) at $x = a$ denoted by $f'(a + 0) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists and

Left hand derivative (LHD) at $x = a$ denoted by $f'(a - 0) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$ also exists.

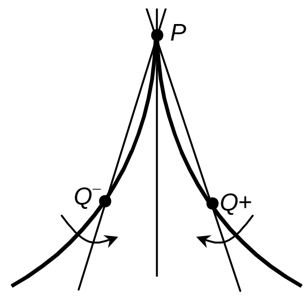
In both these cases, we have assumed $h > 0$.

A function whose graph is otherwise smooth will fail to have a derivative where the graph has

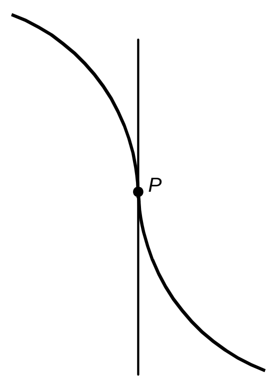
(i) a corner, where the one-sided derivatives differ



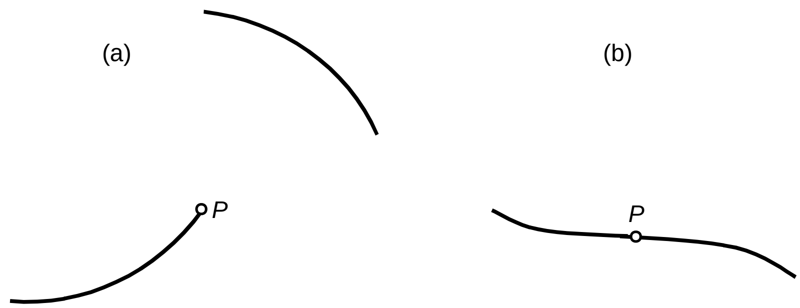
(ii) a cusp, where the slope of *PQ* approaches ∞ from one side and –∞ from the other



(iii) a vertical tangent, where the slope of *PQ* approaches ∞ from both sides or approaches –∞ from both sides (here, –∞)



(iv) a discontinuity



A function is continuous at every point where it has a derivative.

Proof: Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, that $\lim_{h\to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + (f(c+h) - f(c))$$

$$= f(c) + \frac{f(c+h)-f(c)}{h}.h\ .$$

Now take limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\ .\ \lim_{h\to 0} h$$

$$= f(c) + f'(c)\ .\ 0 = f(c) + 0 = f(c).$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Differentiability in an interval

(i) Differentiability in an open interval (a, b)

The function of $y = f(x)$ is said to be differentiable in (a, b) if it is differentiable at each point $x \in (a, b)$

(ii) In an closed interval $[a, b]$

The function $y = f(x)$, is said to be differentiable in $[a, b]$ if $f'(a + 0)$, $f'(b - 0)$ exist and $f'(x)$ exist for all $x \in (a, b)$.

***Illustration* 4**

Question: **Differentiate $e^{\sqrt{x}}$, with respect to x, from first principle.**

Solution: Let $f(x) = e^{\sqrt{x}}$

$\therefore$ $f'(x) = \lim_{h\to 0}\frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h}$, h is small increment in x

$$= \lim_{h\to 0}\frac{e^{\sqrt{x}}[e^{\sqrt{x+h}-\sqrt{x}} - 1]}{h} = \lim_{h\to 0} e^{\sqrt{x}}.\frac{e^{\sqrt{x+h}-\sqrt{x}} - 1}{\sqrt{x+h}-\sqrt{x}}.\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= e^{\sqrt{x}}.1.\lim_{h\to 0}\frac{x+h-x}{h[\sqrt{x+h}+\sqrt{x}]} = e^{\sqrt{x}}.\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

4. DERIVATIVES OF SOME OF THE FREQUENTLY USED FUNCTIONS

Function	Derivative
c (constant)	0
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$
$\text{cosec}^{-1} x$	$\frac{-1}{\|x\|\sqrt{x^2-1}}$
$\log_e x$	$1/x$
$\log_a x$	$(1/x) \log_e a$
x^n	nx^{n-1}
a^x	$a^x \log_e a$
e^x	e^x

The above written derivatives can be easily found by using the definition of differentiation.

5. RULES TO FIND OUT DERIVATIVES

Let *u* and *v* are differentiable functions of '*x*'.

(i) The sum rule

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

e.g. $\frac{d}{dx}(2e^x + 3\log x) = 2\frac{de^x}{dx} + 3\frac{d\ (\log x)}{dx} = 2e^x + \frac{3}{x}$

Illustration 5

Question: **Differentiate $5\sin x - 2\log_e x$.**

***Solution*:** $\frac{d}{dx}(5\sin x - 2\log_e x) = \frac{d}{dx}(5\sin x) - \frac{d}{dx}(2\log_e x) = 5\cos x - \frac{2}{x}$

(ii) Product rule

$$\frac{d\,(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g. $\frac{d\,((\sin x)\, e^x)}{dx} = \sin x\,\frac{de^x}{dx} + e^x\,\frac{d\ (\sin x)}{dx}$

$= (\sin x)\, e^x + (\cos x)\, e^x.$

Illustration 6

Question: **Differentiate $x^2e^x \sin x$.**

***Solution*:** First we differentiate x^2e^x

$$\frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right)$$

$= x^2e^x + 2xe^x$

Now, $\frac{d}{dx}\left(x^2e^x \sin x\right) = x^2e^x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}\left(x^2e^x\right)$

$= x^2e^x \cos x + \sin x\left(x^2 + 2x\right)e^x$

$= e^x\left(x^2\cos x + x^2\sin x + 2x\sin x\right) = xe^x\left(x\cos x + x\sin x + 2\sin x\right)$

(iii) The quotient rule

Here $v(x) \neq 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{v\,du}{dx} - \frac{u\,dv}{dx}}{v^2}$$

e.g. $$\frac{d}{dx}\left(\frac{\tan x}{x}\right)=\frac{x\frac{d(\tan x)}{dx}-(\tan x)\frac{dx}{dx}}{x^2}$$

$$=\frac{x\sec^2 x-\tan x}{x^2}$$

Illustration 7

Question: **Differentiate** $\frac{e^x}{1+\sin x}$.

***Solution*:** $$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right)=\frac{(1+\sin x)\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$=\frac{(1+\sin x)e^x-e^x\cos x}{(1+\sin x)^2}=\frac{e^x(1+\sin x-\cos x)}{(1+\sin x)^2}$$

(iv) Chain rule

The chain rule is probably the most widely used differentiation rule in mathematics. Chain rule says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points.

The formula is $[f(g(x))]'=f'(g(x)).g'(x)$

Illustration 8

Question: **Differentiate** $\sin x^2$.

***Solution*:** Put $y=x^2$ and $z=\sin y$

Then $\frac{dy}{dx}=2x$ and $\frac{dz}{dy}=\cos y$

$$\therefore \frac{d}{dx}(\sin x^2)=\frac{dz}{dx}=\frac{dz}{dy}.\frac{dy}{dx}=(\cos y)\,(2x)=(\cos x^2)(2x)=2x\cos x^2$$

This solution can be rewritten using a more convenient notation in the following manner:

$$\frac{d}{dx}(\sin x^2)=\frac{d(\sin x^2)}{d(x^2)}.\frac{d(x^2)}{dx}=\cos x^2.2x=2x\cos x^2$$

Illustration 9

Question: **Differentiate** $\sin^3 x.\sin 3x$ **w.r.t.** x.

Solution: $\frac{d}{dx}\left(\sin^3 x.\sin 3x\right) = \sin^3 x.\frac{d}{dx}(\sin 3x) + \sin 3x.\frac{d}{dx}(\sin x)^3$

$= \sin^3 x.\cos 3x.3 + \sin 3x.3(\sin x)^2.\cos x$

$= 3\sin^2 x[\sin x\cos 3x + \cos x.\sin 3x]$

$= 3\sin^2 x.\sin(x+3x) = 3\sin^2 x\sin 4x$

6. DERIVATIVES OF IMPLICIT FUNCTIONS

If a relation *x* and *y* is such that *y* cannot be expressed in terms of *x* then *y* is called an implicit function of *x*. The derivative of implicit function can be clear from the given example:

Illustration 10

Question: **If $x + y = \sin(xy)$, find $\frac{dy}{dx}$.**

Solution: Given $x + y = \sin(xy)$

Differentiating with respect to x, we get

$1 + \frac{dy}{dx} = \frac{d}{dx}(\sin(xy))$

$= \frac{d}{dx}\sin(xy).\frac{d}{dx}(xy)$

$= \cos(xy).\left(1.y + x\frac{dy}{dx}\right) = y\cos(xy) + x\cos(xy)\frac{dy}{dx}$

$\therefore \quad [1 - x\cos(xy)]\frac{dy}{dx} = y\cos(xy) - 1$

$\therefore \quad \frac{dy}{dx} = \frac{y\cos(xy) - 1}{1 - x\cos(xy)}$

7. DERIVATIVES OF INVERSE TRIOGONOMETRIC FUNCTIONS

The inverse function are continuous functions in their domain, we apply chain rule to find derivative of these functions.

Illustration 11

Question: **Differentiate the function $\tan^{-1}(\sec x + \tan x)$.**

Solution: Let $y = \tan^{-1}(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}(\sec x + \tan x)\right)$$

$$\frac{1}{1+(\sec x + \tan x)^2}.\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{(1+\sec^2 x + \tan^2 x + 2\sec x \tan x)}(\sec x \tan x + \sec^2 x)$$

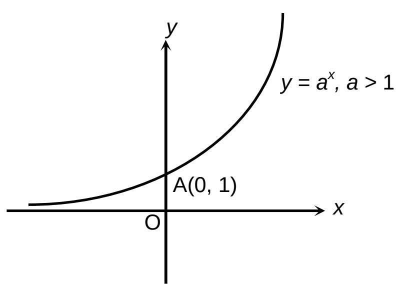
$$= \frac{\sec x \tan x + \sec^2 x}{(2\sec^2 x + 2\sec x \tan x)} = \frac{\sec x(\tan x + \sec x)}{2\sec x(\sec x + \tan x)} = \frac{1}{2}$$

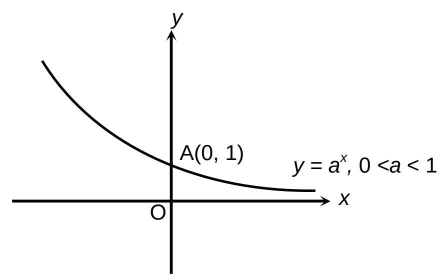
8. EXPONENTIAL FUNCTION

General form of exponential function is $y = a^x$, where a is greater than 0 but not equal to 1.

The domain of function $y = a^x$ is all real number and range is (0, ∞)

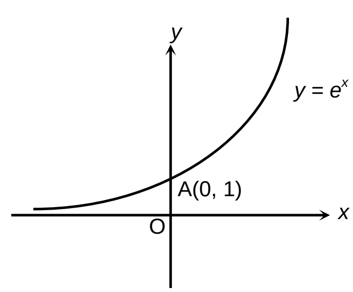
Graph of $y = a^x$ is as given in the figure:





Domain of function $y = e^x$ is R and range is $(0, \infty)$

Graph of $y = e^x$ is as given in the figure:

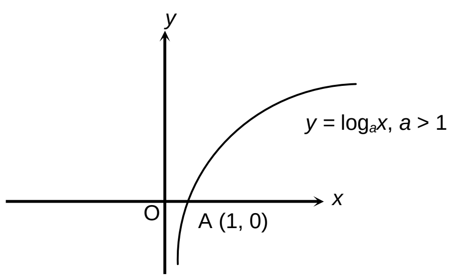


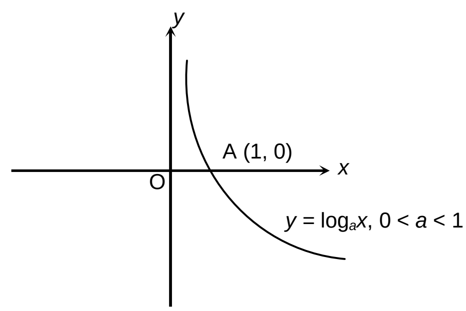
9. LOGARITHMIC FUNCTIONS

If $a > 0$ but $a \neq 1$, then $\forall x \in (0, \infty)$ function $y = \log_a x$ is called logarithmic function.

Domain of $f(x) = (0, \infty)$ and Range of $f(x) = (-\infty, \infty)$

The graph of the function $y = \log_a x$ is as given in the figure:





9.1 Some important formulae

(i) $\log_a mn = \log_a m + \log_a n$

(ii) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(iii) $\log_a p = \frac{\log_b p}{\log_b a}$

(iv) $a^{\log_a x} = x$

Note: (i) $y = \log_e x$ is called natural logarithm function.

(ii) Exponential and logarithm function are inverse of each other.

10. LOGARITHMIC DIFFERENTIATION

If a function is of the form $y = f(x) = [u(x)]^{v(x)}$, then to differentiate take log of both side $\log y = v(x)\log[u(x)]$ and, then differentiate it.

Illustration 12

Question: **If $y = e^{x+e^{x+e^{x+\ldots\text{ to }\infty}}}$, prove that $\frac{dy}{dx} = \frac{y}{1-y}$.**

Solution: Given $y = e^{x+e^{x+e^{x+\ldots\text{ to }\infty}}}$

y $\quad y = e^{x+y}$ …(i)

Taking logarithm, we get $\log y = (x+y)\log_e e$

or $\log y = x + y$ $\quad [\because \log_e e = 1]$

Differentiating with respect to *x*, we get $\frac{1}{y}.\frac{dy}{dx} = 1 + \frac{dy}{dx}$

or $\left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$ or $\left(\frac{1-y}{y}\right)\frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{y}{1-y}$

11. To Find Derivative when Parametric Equation are given

Let $x = f(t)$ and $y = \phi(t)$, be differentiable functions of parameter *t* and $t = \psi(x)$ be the inverse function $x = f(t)$, then $y = \phi[\psi(x)]$ is a function of *x* and *y* is a function of *t* and *t* is function of *x*.

Now $\quad \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$ …(i)

But $\quad \frac{dt}{dx} = \frac{1}{dx/dt}$ …(ii)

$\therefore$ from (i) $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Illustration 13

Question: **If $x = \log t + \sin t$, $y = e^t + \cos t$. Find $\frac{dy}{dx}$.**

Solution: $x = \log t + \sin t$

$\therefore \quad \frac{dx}{dt} = \frac{1}{t} + \cos t \qquad$ ....(i)

and $y = e^t + \cos t$, $\therefore \quad \frac{dy}{dt} = e^t - \sin t \qquad$ ....(ii)

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t - \sin t}{\frac{1}{t} + \cos t} = \frac{t\left(e^t - \sin t\right)}{1 + t\cos t}$

12. SECOND ORDER DERIVATIVE

Derivative of *y* with respect to *x* (it if exists) is denoted by $\frac{dy}{dx}$ and is called the first derivative of *y* or the first order derivative of *y*. It is also denoted by y_1 or y'

Derivative of $\frac{dy}{dx}$ with respect to *x* (if it exists) is denoted by $\frac{d^2y}{dx^2}$ and is called the second derivative of *y* or the second order derivative of *y*. It is also denoted by y_2 or y''

Illustration 14

Question: **If $y = e^{\tan x}$ prove that $\cos^2 x \frac{d^2y}{dx^2} - (1 + \sin 2x)\frac{dy}{dx} = 0$.**

Solution: Given $y = e^{\tan x}$

$\therefore \quad \log y = \tan x \qquad$ ....(i)

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \sec^2 x$ or $\frac{dy}{dx} = y\sec^2 x \qquad$ ...(ii)

or $\quad \cos^2 x \frac{dy}{dx} = y$

Differentiating again with respect to *x*, we get

$\cos^2 x \frac{d^2y}{dx^2} - 2\cos x \sin x \frac{dy}{dx} = \frac{dy}{dx}$ or $\cos^2 x \frac{d^2y}{dx^2} - \left(1 + \sin 2x\right)\frac{dy}{dx} = 0$

13. MEAN VALUE THEOREM

13.1 ROLLE'S THEOREM

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) and further $f(a) = f(b)$, then there is at least one point $x = c$ on the interval (a, b) where $f'(c) = 0$.

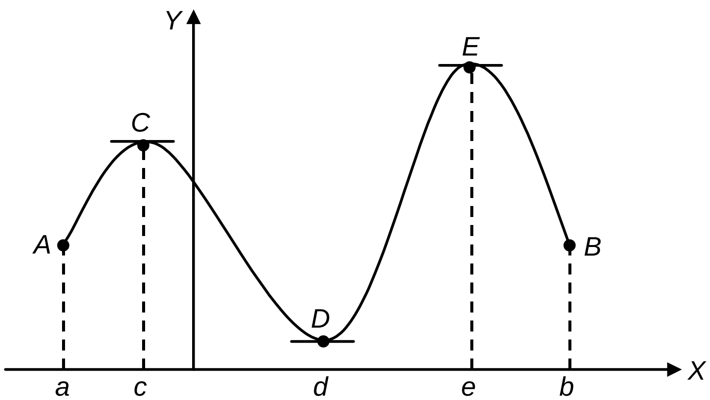


Figure shows the graphical interpretation of Rolle's theorem. The slope of tangent is zero at the points *C*, *D* and *E*.

***Illustration* 15**

Question: **Taking the functions $f(x) = (x$ J $3) \log x$, prove that there is at least one value of x in (1, 3) which satisfies $x \log x = 3$ J x.**

Solution: Given $f(x) = (x - 3) \log x$ …(i)

$\therefore$ $f'(x) = (x - 3)\dfrac{1}{x} + 1 . \log x$ …(ii)

Clearly $f'(x)$ is finite for all positive values of x and hence $f(x)$ is differentiable for all $x > 0$.

$\therefore$ $f(x)$ is differentiable in (1, 3)

$\therefore$ $f(x)$ is also continuous in [1, 3]

Also $f(1) = (1 - 3) \log 1 = 0$ and $f(3) = (3 - 3) \log 3 = 0$

$\therefore$ $f(1) = f(3)$.

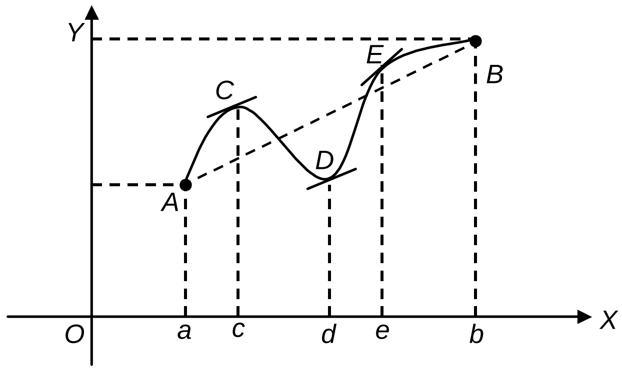
Therefore, by Rolle's theorem, there will be at least one value of x in (1, 3) such that $f'(x)=0$

$\therefore$ from (ii), $\dfrac{x-3}{x} + \log x = 0$

or $x\log x = 3 - x$.

13.2 LAGRANGE'S MEAN VALUE THEOREM

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b), then there exists atleast one point $x = c$ in the interval (a, b) where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



The geometrical meaning is clear from the graph.

$\frac{f(b)-f(a)}{b-a}$ is the slope of the chord *AB*.

The tangents at *C*, *D* and *E* are parallel to this chord. Rolle's theorem is a special case of Lagrange's Mean Value theorem.

Illustration 16

Question: **Find *c* of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].**

Solution: Given $f(x) = 3x^2 + 5x + 7$ …(i)

$\therefore$ $f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again $f'(x) = 6x + 5$ …(ii)

Here $a = 1$, $b = 3$

Now from Lagrange's mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\therefore \quad 6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17$$

or $6c = 12$ $\therefore$ $c = 2$

PRACTICE PROBLEMS

PP1. Is $f(x) = \frac{1}{x}$ continuous in (0, 2) ?

PP2. Is $f(x) = \frac{1}{x} + \frac{1}{|x|}$ continuous in $(-\infty, 0]$?

PP3. If $f(x) = \begin{cases} ax+1 & x \geq 1 \\ x+2 & x < 1 \end{cases}$ is continuous, then find the value of '*a*'.

PP4. If $f(x) = \begin{cases} 1 & ; \text{ If } x \leq 3 \\ ax+b & ; \text{ if } 3 < x < 5 \\ 7 & ; \text{ if } x \geq 5 \end{cases}$ find the value of *a* and *b* for which $f(x)$ is a continuous function.

PP5. Find the derivative of the function $x^2 \cos x$.

PP6. Find the derivative of the function $e^{\sqrt{\tan x}}$.

PP7. Find the derivative of the function $(3x)^{-3/2}$.

PP8. Verify Rolles' theorem for $f(x) = (x - 1)^2 (x - 2)$ in the interval [1, 2].

PP9. Find '*c*' of mean value theorem for $f(x) = \sqrt{x^2 - 4}$, in the interval [2, 3].

PP10. Is $|x + 1|$ is differentiable at $x = -1$?

PP11. If $f(x) = \begin{cases} x^2 + 3x + b & x \geq 1 \\ 2ax + 3 & x < 1 \end{cases}$ is continuous and differentiable, then find out value of *a* and *b*?

PP12. If $f(x) = 3x + x\sin x$, then find the value of $f'(1)$.

PP13. Differentitate the function given in $(\log x)^x + x^{\log x}$.

PP14. Find all the points of discontinuity of *f* defined by $f(x) = |x| - |x+1|$.

PP15. Verify mean value theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval [*a, b*], where *a* = 1 and *b* = 3. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1.**

Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution:

We have, (LHD at $x = 0$) $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$= \lim_{h \to 0} \frac{|h| - |0|}{-h} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1$

and (RHD at $x = 0$) $= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 2$)

So $f(x)$ is not differentiable at $x = 2$

***Example* 2.**

Show that the function $f(x) = \begin{cases} x \sin \frac{1}{x} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$ is continuous at $x = 0$.

Solution:

We have, (LHL at $x = 0$) $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} -h \sin\left(\frac{1}{-h}\right) = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \times$ (an oscillating number between –1 and 1)

(RHL at $x = 0$) $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \times$ (an oscillating number between –1 and 1)

$= 0$

and $f(0) = 0$

Thus $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$.

Hence $f(x)$ is continuous at $x = 0$.

Example 3.

Determine the value of the constant *m* so that the function

$$f(x)=\begin{cases} m(x^2-2x) & , \text{ if } x<0 \\ \cos x & , \text{ if } x\geq 0 \end{cases}$$ **is continuous.**

Solution:

When $x<0$, we have $f(x)=m(x^2-2x)$, which being a polynomial is continuous at each $x < 0$.

When $x>0$, we have $f(x)=\cos x$, which being a cosine function is continuous at each $x > 0$.

So consider the point $x = 0$

We have, (LHL at $x = 0$) $= \lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0} m(x^2-2x)=0$ for all values of m and

(RHL at $x = 0$) $= \lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0}\cos x=1$

Clearly, $\lim\limits_{x\to 0^-} f(x)\neq \lim\limits_{x\to 0^+} f(x)$ for all values of m.

So, $f(x)$ cannot be made continuous for any value of m. In other words, the value of m does not exist for which $f(x)$ can be made continuous.

Example 4.

If $f(x)=\begin{cases} x^2+3x+a & , \text{ for } x\leq 1 \\ bx+2 & , \text{ for } x>1 \end{cases}$ **is everywhere differentiable, find the values of *a* and *b*.**

Solution:

For $x\leq 1$, we have x^2+3x+a i.e., a polynomial and for $x>1$, we have $f(x)=bx+2$, which is also a polynomial.

Since a polynomial function is everywhere differentiable. Therefore $f(x)$ is differentiable for all $x>1$ and for all $x<1$.

Thus, we have to use the differentiability of $f(x)$ at $x = 1$

$\Rightarrow \quad f(x)$ is continuous at $x = 1$

$\Rightarrow \quad \lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$

$\Rightarrow \quad \lim\limits_{x\to 1} x^2+3x+a=\lim\limits_{x\to 1} bx+2=1+3+a \quad \Rightarrow \quad 1+3+a=b+2 \quad \Rightarrow \quad a-b+2=0$ …(i)

Again $f(x)$ is differentiable at $x = 1$

$\Rightarrow$ (LHD at $x = 1$) = (RHD at $x = 1$)

$$\Rightarrow \quad \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} \quad \Rightarrow \quad \lim_{x\to 1}\frac{x^2+3x+a-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{bx-2-a}{x-1}$$

$\Rightarrow \quad \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1} = \lim_{x\to 1}\frac{bx-b}{x-1}$

$\Rightarrow \quad \lim_{x\to 1}(x+4) = \lim_{x\to 1} b$

$\Rightarrow \quad 5 = b$

Putting $b = 5$ in (i), we get $a = 3$

Hence $a = 3, b = 5$

***Example* 5.**

If $f(x) = x^2 \sin\frac{1}{x}$, when $x \neq 0$

$= 0$, when $x = 0$.

Then find the derivative of $f(x)$ at $x = 0$.

Solution:

By definition derivative of (x) at $x = 0$ i.e.,

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0 \qquad \left[\because\ -1 \le \sin\frac{1}{h} \le 1\right]$$

***Example* 6.**

Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Test Whether

(a) $f(x)$ is continuous at $x = 0$

(b) $f(x)$ is differentiable at $x = 0$.

Solution:

(a) $f(0+) = \lim_{x\to 0+} xe^{-2/x}$ since $|x| = x$

$$= \lim_{x\to 0}\frac{x}{e^{2/x}} = 0$$

$f(0-) = \lim_{x\to 0-} x e^0 = \lim_{x\to 0} x = 0$

$f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$

(b) Regarding differentiability

$$f'(0+) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{xe^{-2/x}}{x} = \lim_{x\to 0}\frac{1}{e^{2/x}} = 0$$

$$f'(0-) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

The two values are different. Hence $f(x)$ is not differentiable at $x = 0$.

Example 7:

$f(x) = \begin{cases} x^3 - x^2 + x + 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$ **prove that $f(x)$ is not differentiability at $x = 1$.**

Solution:

RHD $= f'(1+h) = \lim_{h\to 0}\frac{2-2}{h} = 0$

LHD $= f'(1-h) = \lim_{h\to 0}\frac{(1-h)^3-(1-h)^2+(1-h)+1-2}{-h}$

$$= \lim_{h\to 0}\frac{1-h^3-3h^2+3h-1-h^2+2h-h}{-h} = 1$$

So, $f(x)$ is not differentiability at $x = 1$

Example 8:

Check the function $f(x) = \lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}$ for continuity and differentiability at $x = 0$.

Solution:

Let $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}$. Then,

LHL $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1} = \lim_{h\to 0}\frac{(1/e^{1/h}-1)}{(1/e^{1/h}+1)} = \frac{0-1}{0+1} = -1$$

[as $h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow 1/e^{1/h} \to 0$] …(i)

RHL $= \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0}\frac{(1-1/e^{1/h})}{(1+1/e^{1/h})}$$ [Dividing numerator and denominator by $e^{1/h}$]

$$= \frac{1-0}{1+0} = 1$$ [using (i)]

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$. Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

As $\lim\limits_{x\to 0} f(x)$ does not exist, function is neither continuous nor differentiable at $x = 0$.

Example 9.

If $y \text{ Z } e^{x\sin x^3} \text{ H } (\tan x)^x$, find $\frac{dy}{dx}$.

Solution:

Let $u = e^{x\sin x^3}$

$$\frac{du}{dx} = e^{x\sin x^3}\left\{\sin x^3 + x\cdot(\cos x^3)\cdot 3x^2\right\} = e^{x\sin x^3}(\sin x^3 + 3x^3\cos x^3)$$

Let $v = (\tan x)^x$

log v = x log tan x (logarithms are to the base ‘e’. i.e. natural logarithms)

Differentiating w.r.t x,

$$\frac{1}{v}\frac{dv}{dx} = 1\cdot\log\tan x + x\cdot\frac{1}{\tan x}\sec^2 x$$

$$\therefore \quad \frac{dy}{dx} = e^{x\sin x^3}(\sin x^3 + 3x^3\cos x^3) + (\tan x)^x\left(\log\tan x + \frac{x\sec^2 x}{\tan x}\right)$$

Example 10.

If $y \text{ Z } \sin^{\text{J}1} x$, then show that $\text{[}1 \text{ J } x^2\text{[}\frac{d^2y}{dx^2} \text{ J } x\frac{dy}{dx} \text{ Z } 0$.

Solution:

We have, $y = \sin^{-1} x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ [differentiating with respect to x]

$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = 1$

Differentiating both sides with respect to x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = 0$$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ [multiplying both sides by $\sqrt{1-x^2}$]

EXERCISE – I

1. If $f(x)=\begin{cases}\dfrac{\sin 3x}{x} & , \text{ when } x\neq 0\\ 1 & , \text{ when } x=0\end{cases}$. Find whether $f(x)$ is continuous at x = 0.

2. Show that $f(x)=\begin{cases}1+x^2 & , \text{ if } 0\le x\le 1\\ 2-x & , \text{ if } x>1\end{cases}$ is discontinuous at x = 1.

3. If $f(x)=\begin{cases}2x^2+k & , \text{ if } x\ge 0\\ -2x^2+k & , \text{ if } x<0\end{cases}$, then what should be the value of k so that $f(x)$ is continuous at x = 0.

4. The function $f(x)=\begin{cases}x^2/a & , \text{ if } 0\le x<1\\ a & , \text{ if } 1\le x<\sqrt{2}\\ \dfrac{2b^2-4b}{x^2} & , \text{ if } \sqrt{2}\le x<\infty\end{cases}$ is continuous on $[0,\infty)$, then find the most suitable values of a and b.

5. Find the values of a and b so that the function $f(x)$ defined by

$$f(x)=\begin{cases}x+a\sqrt{2}\sin x & , \text{ if } 0\le x<\dfrac{\pi}{4}\\ 2x\cot x+b & , \text{ if } \dfrac{\pi}{4}\le x<\dfrac{\pi}{2}\\ a\cos 2x-b\sin x & , \text{ if } \dfrac{\pi}{2}\le x\le \pi\end{cases}$$ becomes continuous on $[0,\pi]$.

6. Show that $f(x)=x^{1/3}$ is not differentiable at x = 0.

7. For the function f given by $f(x)=x^2-6x+8$, prove that $f'(5)-3f'(2)=f'(8)$.

8. Write an example of a function which is everywhere continuous but fails to be differentiable exactly at five points.

9. Show that the derivative of the function f given by $f(x)=2x^3-9x^2+12x+9$ at $x=1$ and $x=2$ are equal.

10. Discuss the continuity and differentiability of $f(x)=\begin{cases}(x-c)\cos\left(\dfrac{1}{x-c}\right) & , \quad x \neq c \\ 0 & , \quad x = c\end{cases}$.

11. Differentiate the function with respect to x : $\sin^{-1}(2x^2-1)$, $0 < x < 1$.

12. If $x^2+2xy+y^3=42$ find $\dfrac{dy}{dx}$.

13. Differentiate the following functions with respect to x:

(i) x^x (ii) $x^{\sin x}$

14. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, prove that $y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

15. (i) If $y = 2\sin x + 3\cos x$, show that $\dfrac{d^2y}{dx^2} + y = 0$.

(ii) If $y = \dfrac{\log x}{x}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2\log x - 3}{x^3}$.

EXERCISE – II

1. Show that the function $f(x)$ given by $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x & x\neq 0\\ 2 & x=0\end{cases}$ is continuous at $x = 0$.

2. Discuss the continuity of the function given by $f(x)=|x-1|+|x-2|$ at $x=1$ and $x=2$.

3. If the function $f(x)$ given by $f(x)=\begin{cases}3ax+b & , \text{ if } x>1\\ 11 & , \text{ if } x=1\\ 5ax-2b & , \text{ if } x<1\end{cases}$ is continuous at $x = 1$, find the values of a and b.

4. For what choice of a and b is the function $f(x)=\begin{cases}x^2 & , x\le c\\ ax+b & , x>c\end{cases}$ is differentiable at $x = c$.

5. Discuss the continuity and differentiability of $f(x)=\begin{cases}1-x & , x<1\\ (1-x)(2-x) & , 1\le x\le 2\\ 3-x & , x>2\end{cases}$

6. Show that the function $f(x)=\begin{cases}\dfrac{x}{|x|} & ; x\neq 0\\ 1 & ; x=0\end{cases}$ is discontinuous at $x = 0$.

7. Show that $f(x)=\begin{cases}\dfrac{x^2}{2} & ; \text{ if } 0\le x\le 1\\ 2x^2-3x+\dfrac{3}{2} & ; \text{ if } 1<x\le 2\end{cases}$ is continuous at $x = 1$.

8. Prove that $f(x)=\begin{cases}\sin\dfrac{1}{x} & ; x\neq 0\\ 0 & ; x=0\end{cases}$ is discontinuous at $x = 0$.

9. Prove that $f(x)=\begin{cases}x & , 0\le x<\dfrac{1}{2}\\ 1/2 & , x=\dfrac{1}{2}\\ 1-x & , \dfrac{1}{2}<x\le 1\end{cases}$ is continuous at $x=\dfrac{1}{2}$.

10. Prove that $f(x)=\begin{cases}-x & , & x<0\\ 1 & , & x=0\\ x & , & x>0\end{cases}$ is discontinuous at x = 0.

11. Show that $f(x)=[x]$ is not continuous at x = n, where n is an integer and [**.**] is greatest integer function.

12. Is the function $f(x)=\begin{cases}\dfrac{1-x^n}{1-x} & , & x\neq 1\\ n-1 & , & x=1\end{cases}$, $n\in N$ continuous at x = 1?

13. Find A and B so that $y = A\sin 3x + B\cos 3x$ satisfies the equation
$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=10\cos 3x$.

14. Prove that $f(x)=\begin{cases}-x & , & x<0\\ x^2 & , & 0\le x\le 1\\ x^3-x+1 & , & x>1\end{cases}$ is
(i) not differentiable at x = 0 (ii) differentiable at x = 1

13. Prove that $f(x)=\frac{x}{1+|x|}$ is differentiable for all real x.

14. Verify Lagrange's mean value theorem for the function $f(x) = x(x-1)(x-2)$ in the interval $\left[0,\frac{1}{2}\right]$.

15. Find the value of c of the Largange's mean value theorem for the function $f(x)=x^2-3x+2$ in [1, 2].

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. Yes

PP2. No

PP3. 2

PP4. $a = 3$, $b = -8$

PP5. $-x^2 \sin x + 2x \cos x$

PP6. $\frac{\sec^2 x}{2\sqrt{\tan x}} e^{\sqrt{\tan x}}$

PP7. $-\frac{9}{2}(3x)^{-\frac{5}{2}}$

PP9. $\sqrt{5}$

PP10. No

PP11. $a = 5/2$ and $b = 4$

PP12. $3 + \cos 1 + \sin 1$

PP13. $(\log x)^{x-1}[1 + \log x . \log(\log x)] + 2x^{\log x - 1}.\log x$

PP14. no point of discontinuity

EXERCISE – I

1. Discontinuity

3. k is any real number

4. $a = -1, b = 1$ or $a = 1, b = 1 \pm \sqrt{2}$

5. $a = \frac{\pi}{6}, b = -\frac{\pi}{12}$

10. Not differentiable at $x = c$

11. $\frac{2}{\sqrt{1-x^2}}$

12. $-\frac{2(x+y)}{(2x+3y^2)}$

13. (i) $x^x(1+\log x)$ (ii) $x^{\sin x}\left\{\cos x.\log x + \frac{\sin x}{x}\right\}$

EXERCISE – II

2. Continuous at $x = 1$ & $x = 2$

3. $a = 3$, $b = 2$

4. $a = 2c$, $b = -c^2$

5. Continuous & differentiable at $x = 1$
Discontinuous & Non-differentiable at $x = 2$.

12. Discontinuous

13. $A = \frac{2}{3}, B = -\frac{1}{3}$

15. $\frac{3}{2}$