LESSON 4

DETERMINANTS

1. DETERMINANTS

1.1 EVALUATION OF DETERMINANTS

Determinants of second order: The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ consisting of 2^2 numbers called elements, arranged in two rows and two columns, is called a determinant of second order. The elements a_1 and b_2 are said to lie along the principal diagonal; the elements a_2 and b_1 are said to lie along the secondary diagonal.

The value of the determinant is obtained by forming the product of the elements along the principal diagonal and subtracting from it the product of the elements along the secondary diagonal.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ ... (i)

Determinants of third order: The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 3^2 elements arranged in three rows and three columns, is called a determinant of third order. Its value is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

This may be written as $a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$

or $$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

We can therefore write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad \text{... (ii)}$$

Note that each term of a second order determinant is the product of two quantities and each term of a third order determinant is the product of three quantities.

1.2 MINORS

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element occurs.

For example, the minor of a_1 in (ii) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, and the minor of b_2 in (ii) is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

The minor of a_1 in (i) is b_2 and b_2 may be considered a determinant of first order. Similarly, the minor of a_2 is b_1.

1.3 COFACTORS

In (ii), the elements a_1, b_1, c_1 are multiplied by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

These expressions are called the cofactors of the elements a_1, b_1, c_1.

Generally, the cofactor of an element is its minor with its sign or opposite sign prefixed in accordance with the following rule:

For any determinant if a_{ij} be the element at the intersection of the ith row and jth column, then the cofactor of a_{ij} has positive sign or negative sign before minor of a_{ij} according as $i + j$ is even or odd. The determinant may be expanded along any chosen row or column.

The cofactors of the elements a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 will be denoted by A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 respectively.

For example, element b_3 in (ii) lies at the intersection of the third row and the second column. Since 3 + 2 = 5 is an odd number, we have

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The cofactor B_2 of the element b_2 is $+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, because element b_2 lies at the intersection of the second row and the second column, and 2 + 2 = 4 is an even number.

Let the determinant (ii) be denoted by Δ. When the cofactors are used, the expansion of the determinant takes the compact form:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3.$$

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3$$

and $a_2A_1 + b_2B_1 + c_2C_1 = 0 = a_2A_3 + b_2B_3 + c_2C_3$ etc.

Illustration 1

***Question*:** **Evaluate the determinant**

$$a = \begin{vmatrix} 2 & 3 & 4 \\ 5 & J2 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

***Solution*:** Expanding along the second row, we have

$$\Delta = -5\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$= -5\,(9 - 8) - 2\,(6 - 4) - 1\,(4 - 3)$

$= -5 - 4 - 1 = -10.$

Expanding along the third column, we have

$$\Delta = 4\begin{vmatrix} 5 & -2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$$

$= 4\,(10 + 2) - 1\,(4 - 3) + 3\,(-4 - 15)$

$= 48 - 1 - 57 = -10.$

2. PROPERTIES OF DETERMINANTS

1. If two rows in a determinant are interchanged, the sign of the determinant changes. For example, by interchanging the two rows of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, we get the determinant $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$

But we have $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

2. If the numbers in one row are added, m times the numbers in another row, the value of the determinant remains unaltered.

For example, $\begin{vmatrix} a_1+ma_2 & b_1+mb_2 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

This rule can be extended to more number of rows for higher order determinants.

3. If rows and columns are interchanged, the value of the determinant remains unaltered.

For example, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Another way of saying this is that it makes no difference if we reflect the numbers of the determinant in the line of the principal diagonal. This means that any statement that can truly be made about rows in particular results (1) and (2) can equally well be made about columns.

4. If all the numbers in any row are zeros, the value of the determinant is zero.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

5. If two rows are identical, the value of the determinant is zero.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

6. If the elements of a row are multiplied by any number m, the determinant is multiplied by *m*.

For example, $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

7. Row-Column Operations : The value of determinant remains unchanged when any row (or column) is multiplied by a number or any expression and then added or subtracted from any other row (or column).

i.e. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ma_2 - na_3 & a_2 & a_3 \\ b_1 + mb_2 - nb_3 & b_2 & b_3 \\ c_1 + mc_2 - nc_3 & c_2 & c_3 \end{vmatrix}$

The above operation is written as $C_1 \to C_1 + mC_2 - nC_3$ means C_1 is replaced by $C_1 + mC_2 - nC_3$.

Illustration 2

***Question*:** **Show that**

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a \text{ J } b)(b \text{ J } c)(c \text{ J } a).$$

***Solution*:** Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

If *b* is put equal to a, two rows are exactly alike.

$\therefore$ $\Delta = 0$ when $b = a$

$\therefore$ $(a - b)$ is a factor of Δ (This follows from the factor theorem which states that for $f(x)$, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$).

Similarly ($b - c$) and (c – a) are factors.

Again, Δ is of the third degree in a, b and c.

And we know already three linear factors ($a - b$), ($b - c$) and ($c - a$). If there is another factor, it must be a mere number.

Thus $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = N\,(a - b)\,(b - c)\,(c - a)$, where N is a number.

By equating coefficients of bc^2 on both sides, $N = 1$

$\therefore\ \Delta = (a - b)\,(b - c)\,(c - a)$.

Alternative method:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting the second row from the first and then the third row from the second, we have

$$\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (\mathrm{a} - b)\,(b - c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Now expanding along the first column, we have

$\Delta = (a - b)\,(b - c)\,[(b + c) - (a + b)] = (a - b)\,(b - c)\,(c - a)$.

***Illustration* 3**

***Question*:** **Show that**

$$\mathrm{a} = \begin{vmatrix} \boldsymbol{b}\mathrm{H}\boldsymbol{c} & \boldsymbol{c} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{c}\mathrm{H}\boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b}\mathrm{H}\boldsymbol{a} \end{vmatrix} = \mathbf{4}\boldsymbol{abc}.$$

***Solution*:**

$$\Delta = \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \text{ by } R_1 : R_1 + R_2 + R_3$$

Now take 2 as a common factor and then apply $R_2 : R_2 - R_1$ and $R_3 : R_3 - R_1$

$$\Delta = 2\begin{vmatrix} b+c & c+a & a+b \\ -b & 0 & -b \\ -c & -c & 0 \end{vmatrix}$$

Now apply $C_2 : C_2 - C_1$

$$\Delta = 2\begin{vmatrix} b+c & a-b & a+b \\ -b & b & -b \\ -c & 0 & 0 \end{vmatrix}$$

Now expand through R_3

$\Delta = 2[(-c)\{-ab + b^2 - ab - b^2\}] = 4abc$

Illustration 4

***Question*:** **Show that** $\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2-bc} & \boldsymbol{b^2-ca} & \boldsymbol{c^2-ab} \end{vmatrix} = \mathbf{0}$.

***Solution*:** $C_1 : C_1 - C_2$ and $C_2 : C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2+c(a-b) & b^2-c^2+a(b-c) & c^2-ab \end{vmatrix}$$

i.e., $(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b+c & b+c+a & c^2-ab \end{vmatrix}$

i.e., $(a-b)(b-c)[(b+c+a)-(a+b+c)] = 0$.

Note: If a determinant can be so transformed that two elements in a row or column are made zero, then the determinant can be expanded in terms of that row or column.

Illustration 5

***Question*:** **Show that** $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \boldsymbol{bc} & \boldsymbol{b+c} \\ \mathbf{1} & \boldsymbol{ca} & \boldsymbol{c+a} \\ \mathbf{1} & \boldsymbol{ab} & \boldsymbol{a+b} \end{vmatrix}$.

***Solution*:** We have $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & bc & a+b+c-a \\ 1 & ca & a+b+c-b \\ 1 & ab & a+b+c-c \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & a+b+c \\ 1 & ca & a+b+c \\ 1 & ab & a+b+c \end{vmatrix} - \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} + \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}$$

$$= \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}$$, since the first determinant vanishes

$$= \frac{1}{abc}\begin{vmatrix} abc & a & a^2 \\ abc & b & b^2 \\ abc & c & c^2 \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

3. AREA OF A TRIANGLE

In earlier classes, we have studied that the area of triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by the expression $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \ldots(i)$$

Remarks

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (i).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

***Illustration* 6**

***Question*:** **Find the area of the triangle whose vertices are (3, 8), (–4, 2) and (5, 1).**

***Solution*:** The area of triangle is given by $\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

$$= \frac{1}{2}[3(2-1) - 8(-4-5) + 1(-4-10)] = \frac{1}{2}(3 + 72 - 14) = \frac{61}{2}$$

***Illustration* 7**

***Question*:** **Find the equation of the line joining *A*(1, 3) and *B*(0, 0) using determinants and find *k* if *D*(*k*, 0) is a point such that area of triangle *ABD* is 3 sq. units.**

***Solution*:** Let $P(x, y)$ be any point on AB.

Then the area of triangle ABP is zero.

So, $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$

This gives $\frac{1}{2}(y - 3x) = 0$ or $y = 3x$, which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives, $\frac{-3k}{2} = \pm 3$, i.e., $k = \mp 2$

4. ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n \times n}$ be any $n \times n$ matrix. The transpose B' of the matrix $B = [C_{ij}]_{n \times n}$, where C_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$, is called the adjoint of the matrix A and is denoted by the symbol *adj A*.

***Illustration* 8**

Question: **If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then find *adj* A.**

Solution: In $|A|$, the cofactor of α is δ and the cofactor of β is $-\gamma$. Also the cofactor of γ is $-\beta$ and the cofactor of δ is α. Therefore the matrix B formed of the cofactor of the elements of $|A|$ is

$$B = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$$

Now Adj A = the transpose of the matrix B = $\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

5. INVERSE OR RECIPROCAL OF A MATRIX

Let A be any n-rowed square matrix. Then a matrix B, if it exists, such that $AB = BA = I_n$ is called inverse of A.

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

If A be an invertible matrix, then the inverse of A is $\frac{1}{|A|}$ Adj. A. It is usual to denote the inverse of A by A^{-1}.

5.1 PROPERTIES

(i) $(AB)^{-1} = B^{-1}A^{-1}$, (ii) $(A')^{-1} = (A^{-1})'$ (iii) $(A^{-1})^{\theta} = (A^{\theta})^{-1}$, $\theta \in R$

Illustration 9

Question: **Find the inverse of the matrix** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: We have

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}, \text{ applying } C_3 \to C_3 - 2C_2$$

$$= -1\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}, \text{ expanding the determinant along the first}$$

$= -2$

Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

i.e. , are –1 , 8 , –5 respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

i.e. , are 1, –6, 3 respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

i.e. , are –1, 2, –1 respectively.

Therefore the Adj. A = the transpose of the matrix B where

$$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \quad \therefore \text{ Adj. } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}$ Adj. A and here $|A| = -2$.

$$\therefore \quad A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

6. SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be non-singular or singular according as $|A| \neq 0$ or $|A| = 0$.

7. SOLUTION OF SYSTEM OF LINEAR EQUATIONS USING INVERSE OF A MATRIX

Let us express the system of linear equations as matrix equations and solve them using inverse of the matrix.

Consider the system of equations

$a_1x + b_1y + c_1z = d_1$,

$a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the system of equations can be written as $AX = B$, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I:

If A is a nonsingular matrix, then its inverse exists.

Now $AX = B$

or $A^{-1}(AX) = A^{-1}B$ or $(A^{-1}A)X = A^{-1}B$

or $IX = A^{-1}B$ or $X = A^{-1}B$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II:

If A is a singular matrix, then $|A| = 0$

In this case, we calculate $(adj\ A)B$

If $(adj\ A)B \neq O$, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If $(adj\ A)B = O$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Note: If a system of equations has at least one solution, then system is called consistent.

If a system of equations has no solution, then system is called inconsistent.

***Illustration* 10**

Question: **Solve the following system of equations by matrix method**

$$3x - 2y + 3z = 8,\ 2x + y - z = 1 \text{ and } 4x - 3y + 2z = 4.$$

Solution: Given the equations in the form $AX = B$, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B - \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

Now, $|A| = 3(2-3) + 2(4+4) + 3(-6-4) = -17 \neq 0$

Hence A is non-singular and so its inverse exists.

Now $A_{11} = -1,\ A_{12} = -8,\ A_{13} = -10$

$A_{21} = -5,\ A_{22} = -6,\ A_{23} = 1$

$A_{31} = -1,\ A_{32} = 9,\ A_{33} = 7$

Therefore $A^{-1} = -\dfrac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

So $X = A^{-1}B = -\dfrac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\dfrac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence $x = 1,\ y = 2$ and $z = 3$

PRACTICE PROBLEMS

PP1. Evaluate the determinant
$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

PP2. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 3 \\ x & 4 & 5 \end{vmatrix} = 0$

Then what is the value of x?

PP3. Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

PP4. Using cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

PP5. If the equation $x = ay + z$, $y = az + x$ and $z = ax + y$ are the consistent having non-trivial solution, then prove that $a^3 + 3a = 0$.

PP6. If $f(x) = \begin{vmatrix} x^2 - x & x^3 & x^4 - 1 \\ 2x - 1 & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{vmatrix}$, then find the coefficient of x in $f(x)$.

PP7. If the system of equations
$3x + 10y + 17z = 0$, $x + 6y + 13z = 0$ and $20x - 13y + \lambda z = 0$ has a non-trivial solution then find the solution.

PP8. If $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$, show that $A^2 + 3A + 4I_2 = O$ and hence find A^{-1}.

PP9. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

PP10. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the values of a and b such that $A^2 + aA + bI = O$. Hence find A^{-1}.

PP11. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -3 & 0 \end{bmatrix}$, verify that $adj(AB) = (adj\ B)(adj\ A)$

PP12. Using matrices solve the equations $2x + 5y = 1$, $3x + 2y = 7$.

PP13. Using matrices solve the equations $3x - y + z = 5$, $2x - 2y + 3z = 7$ and $x + y - z = -1$.

PP14. If area of triangle is 35 sq. units with vertices (2, –6), (5, 3) and (k, 4). Find the value of k.

PP15. Show that the points (a, 0), (0, b) and (1, 1) are collinear, if $\frac{1}{a} + \frac{1}{b} = 1$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Show that

$$\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0 .$$

Solution:

Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$ we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix} = 0 \qquad [\because \ C_1 \text{ consists of all zeros}]$$

***Example* 2:**

Show that

$$\begin{vmatrix} \boldsymbol{b+c} & \boldsymbol{a-b} & \boldsymbol{a} \\ \boldsymbol{c+a} & \boldsymbol{b-c} & \boldsymbol{b} \\ \boldsymbol{a+b} & \boldsymbol{c-a} & \boldsymbol{c} \end{vmatrix} = \boldsymbol{3abc - a^3 - b^3 - c^3} .$$

Solution:

Let $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = \begin{vmatrix} b+c & -b & a \\ c+a & -c & b \\ a+b & -a & c \end{vmatrix}$ Applying $C_2 \to C_2 - C_3$

$= -\begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix}$ Applying $C_1 \to C_1 + C_2$

$= [a^3 + b^3 + c^3 - 3abc]$ by circulate determinant.

***Example* 3:**

Without expanding the determinant at any stage, show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B,$$

where A and *B* are determinants of third order independent of *x*.

***Solution*:**

Let Δ stand for the given determinant. Let us add the elements of the third row to the corresponding elements of the first row.

Then $\Delta = \begin{vmatrix} 2x^2+3x+3 & 3x & 3x-3 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

$$= \begin{vmatrix} 2x^2+3x & 3x & 3x-3 \\ 2x^2+3x & 3x & 3x-3 \\ x^2+2x & 2x-1 & 2x-1 \end{vmatrix} + \begin{vmatrix} 3 & 3x & 3x-3 \\ -1 & 3x & 3x-3 \\ 3 & 2x-1 & 2x-1 \end{vmatrix}$$

The first determinant vanishes since two rows are identical.

$$\therefore \quad \Delta = \begin{vmatrix} 3 & 3x & 3x-3 \\ -1 & 3x & 3x-3 \\ 3 & 2x-1 & 2x-1 \end{vmatrix}$$

Subtracting the third column from the second column, we have

$$\Delta = \begin{vmatrix} 3 & 3 & 3x-3 \\ -1 & 3 & 3x-3 \\ 3 & 0 & 2x-1 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 3x \\ -1 & 3 & 3x \\ 3 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 3 & 3 & -3 \\ -1 & 3 & -3 \\ 3 & 0 & -1 \end{vmatrix} = x\begin{vmatrix} 3 & 3 & 3 \\ -1 & 3 & 3 \\ 3 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 3 & -3 \\ -1 & 3 & -3 \\ 3 & 0 & -1 \end{vmatrix}$$

***Example* 4:**

Solve for *x*:

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$$

***Solution*:**

Let $\quad \Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix}$

Subtracting twice the first row from the third row, we have $\Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 1 & -4 & 0 \end{vmatrix}$

Now subtracting $\frac{3}{2}$ times the first row from the second row, we have $\Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 2 & 0 & -\frac{3}{2} \\ 1 & -4 & 0 \end{vmatrix}$

Now adding 4 times the first column to the second column, we have

$$\Delta = \begin{vmatrix} 4x & 22x+2 & 8x+1 \\ 2 & 8 & -\frac{3}{2} \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along the third row, we have

$$\Delta = -\frac{3}{2}(22x + 2) - 8(8x + 1) \quad = - 33x - 3 - 64x - 8 = - 97x - 11$$

The given equation now becomes $- 97x - 11 = 0$ or $x = -\frac{11}{97}$

Example 5:

If *a*, *b*, *c* are unequal and

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0, \text{ find the value of } abc.$$

Solution:

Let $\Delta = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$

Then $\Delta = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

The second determinant = $abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$

$$\therefore \Delta = (1 + abc)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

= (1 + abc) (a – b) (b – c) (c – a)

$\therefore$ Δ = 0 means that (1 + abc) (a – b) (b – c) (c – a) = 0

Since a, b, c are all unequal, none of the factors $a - b$, $b - c$, $c - a$ can be zero.

$\therefore$ we have 1 + abc = 0 or abc = – 1

***Example* 6:**

Without expanding evaluate the determinant

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 & (a^z - a^{-z})^2 & 1 \end{vmatrix}$$ **, where $a > 0$ and $x, y, z \in R$.**

***Solution*:**

Let Δ be the given determinant. Then applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (a^y - a^{-y})^2 & 1 \\ 4 & (a^z - z^{-z})^2 & 1 \end{vmatrix}$$ [using $(a+b)^2 - (a-b)^2 = 4ab$]

$$= \begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (a^y - a^{-y})^2 & 1 \\ 1 & (a^z - a^{-z})^2 & 1 \end{vmatrix}$$ (taking out 4 common from C_1)

$= 4 \times 0 = 0$ [$\because$ C_1 and C_2 are identical]

***Example* 7:**

Use matrix method to solve the following system of equations:

$$x - 2y - 4 = 0,\ -3x + 5y + 7 = 0.$$

***Solution*:**

Given the equations $x - 2y = 4$

$$-3x + 5y = -7$$

or $\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix}$$

and $B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

Now $|A| = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = 5 - 6 = -1 \neq 0$

So, the given system has a unique solution, given by $X = A^{-1}B$.

Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then

$C_{11} = (-1)^{1+1}5 = 5,\ C_{12} = (-1)^{1+2}(-3) = 3,\ C_{21} = (-1)^{2+1}(-2) = 2$

and $C_{22} = (-1)^{2+2}1 = 1$

$\therefore \quad adj\ A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

So $\quad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{(-1)}\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

$\therefore \quad X = A^{-1}B$

$\Rightarrow \quad X = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -20 & 14 \\ -12 & 7 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$\Rightarrow$ $x = -6$ and $y = -5$

Hence $x = -6$ and $y = -5$ is the required solution.

***Example* 8:**

Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ **given that $x > 0, y > 0, z > 0$.**

***Solution*:**

The determinant is equal to

$$\begin{vmatrix} 1 & \dfrac{\log_{10} y}{\log_{10} x} & \dfrac{\log_{10} z}{\log_{10} x} \\ \dfrac{\log_{10} x}{\log_{10} y} & 1 & \dfrac{\log_{10} z}{\log_{10} y} \\ \dfrac{\log_{10} x}{\log_{10} z} & \dfrac{\log_{10} y}{\log_{10} z} & 1 \end{vmatrix} = \frac{1}{\log_{10} x} \cdot \frac{1}{\log_{10} y} \cdot \frac{1}{\log_{10} z} \begin{vmatrix} \log_{10} x & \log_{10} y & \log_{10} z \\ \log_{10} x & \log_{10} y & \log_{10} z \\ \log_{10} x & \log_{10} y & \log_{10} z \end{vmatrix}$$

$$= \frac{1}{\log_{10} x \cdot \log_{10} y \cdot \log_{10} z} \times 0 \quad = 0.$$

***Example* 9:**

Solve the following system of homogeneous equations:

$2x$ H $3y$ J z Z 0, x J y J $2z$ Z 0 and $3x$ H y H $3z$ Z 0 .

***Solution*:**

The given equation $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or $AX = O$, where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = 2(-3+2) - 3(3+6) - 1(1+3)$

$$= -2 - 27 - 4 = -33 \neq 0$$

Thus $|A| \neq 0$

So, the given system has only the trivial solution given by $x = y = z = 0$.

***Example* 10:**

Prove that $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$

***Solution*:**

$$\Delta = a_1a_2a_3\begin{vmatrix} \frac{1}{a_1}+1 & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix} \text{ by } C_1 : C_1 \to \frac{1}{a_1}C_1 \text{ etc.}$$

$$= a_1a_2a_3\begin{vmatrix} \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix} \text{ by } C_1 : C_1+C_2+C_3$$

$$= a_1a_2a_3\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1\right)\begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 1 & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ 1 & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix} \text{ by taking out a common factor from } C_1$$

$$= a_1a_2a_3\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1\right)\begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \text{ by } R_2: R_2 - R_1 \text{ and } R_3: R_3 - R_1$$

EXERCISE – I

1. If $x=-9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}=0$, find other roots.

2. Without expanding show that the value of each determinant is zero.

(i) $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$ (ii) $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

3. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}=2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

4. Show that $\begin{vmatrix} a+b+c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}=2(a+b+c)^3$.

5. Show that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}=(a^3-1)^2$.

6. Show that $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$.

7. Show that $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix}=(x-y)(y-z)(z-x)$.

8. Find the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ {}^{m}C_1 & {}^{(m+1)}C_1 & {}^{(m+2)}C_1 \\ {}^{m}C_2 & {}^{(m+1)}C_2 & {}^{(m+2)}C_2 \end{vmatrix}$.

9. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}=x+iy$, then find the value of x, y.

10. If the system of equation $x+2y-3z=1,\ (a+2)z=3,\ (2a+1)y+z=2$ is inconsistent, then find the value of *a*.

11. Solve the equations $x+2y+z=7,\ x+3z=11,\ 2x-3y=1$.

12. If $A=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$, show that $A^2-4A-I_2=O$. Hence find A^{-1}.

13. If $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$, show that $A'A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$.

14. Compute the adjoint of the matrix $A=\begin{bmatrix}3 & 4\\ 5 & 7\end{bmatrix}$ and verify that $(adj\ A)A=|A|\,I$.

15. Show that $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ satisfies the equation $x^2-6x+17=0$. Hence find A^{-1}.

EXERCISE – II

1. Find the value of the determinant $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$.

2. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

3. Solve the following equations, using matrix method

$x - y + 3z = 6$

$x + 3y - 3z = -4$

$5x + 3y + 3z = 10$.

4. The sum of three numbers is 6. If we multiply third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12, use matrix to find the numbers.

5. Given that $x = cy + bz$, $y = az + cx$, $z = bx + ay$; where *x, y, z* are not all zero, show that $a^2 + b^2 + c^2 + 2abc = 1$.

6. The roots of $\begin{vmatrix} 2x+4 & 3+4x & 11 \\ 16 & 12+2x & 4x+2 \\ 7 & 6 & 2x \end{vmatrix}$ = 0 are

7. If a_1 a_2 …. form a G. P. and $a_i > 0$ for all $i \geq 1$, then $\Delta = \begin{vmatrix} \log a_m & \log a_{m+1} & \log a_{m+2} \\ \log a_{m+3} & \log a_{m+4} & \log a_{m+5} \\ \log a_{m+6} & \log a_{m+7} & \log a_{m+8} \end{vmatrix}$ is

8. The value of the determinant $\begin{vmatrix} \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) \\ \log_b\left(\frac{y}{z}\right) & \log_b\left(\frac{z}{x}\right) & \log_b\left(\frac{x}{y}\right) \\ \log_c\left(\frac{z}{x}\right) & \log_c\left(\frac{x}{y}\right) & \log_c\left(\frac{y}{z}\right) \end{vmatrix}$ is equal to

9. Show that the following system of equations is consistent $2x - y + 3z = 5, 3x + 2y - z = 7$ and $4x + 5y - 5z = 9$. Also, find the solution.

10. Solve the equation using matrix-method:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

11. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, find A^{-1}, hence solve the equations $x + y + 2z = 0, x + 2y - z = 9,$ $x - 3y + 3z = -14$.

12. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, find AB and use this result to solve the equations $2x - y + z = -1, -x + 2y - z = 4, x - y + 2z = -3$.

13. Solve the equations

$$\lambda x + 2y - 2z - 1 = 0 , 4x + 2\lambda y - z - 2 = 0 , 6x + 6y + \lambda z - 3 = 0 ,$$

considering specially the case when $\lambda = 2$.

14. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

15. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs. 60. The cost of 2 kg onion, 4kg wheat and 6 kg rice is Rs. 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs. 70. Find cost of each item per kg by matrix method.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. zero

PP2. $x = -9$

PP3. $\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

PP4. 7

PP6. 6

PP7. $x = k$, $y = -2k$, $z = k$

PP8. $\begin{bmatrix} -1/2 & 1/4 \\ -1/2 & -1/4 \end{bmatrix}$

PP10. $a = -4$, $b = 1$ and $A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

PP12. $x = 3$, $y = -1$

PP13. $x = 1, y = -1, z = 1$

PP14. 12, –2

EXERCISE – I

1. 2, 7

8. 1

9. $x = 0$, $y = 0$

10. $-\frac{1}{2}$

11. $x = 2$, $y = 1$, $z = 3$

12. $A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

14. $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$

15. $\frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

EXERCISE – II

1. a^3

3. $x = 1 - k,\ y = k;\ z = \dfrac{2k+5}{3},\ k \in R$

4. 3, 1, 2

6. $\left(-\dfrac{9}{2}, 1, \dfrac{7}{2}\right)$

7. zero

8. zero

9. $x = \dfrac{17-5k}{7},\ y = \dfrac{11k-1}{7}$ and $z = k$, where k is any real number satisfy the given system of equations.

10. $x = 2$, $y = 3$, $z = 5$

11. $A^{-1} = \dfrac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix};\ x = 1,\ y = 3,\ z = -2$

12. $AB = 4I,\ x = 1,\ y = 2,\ z = -1$

13. $\left.\begin{array}{l} x = 1/2 - c \\ y = c \\ z = 0 \end{array}\right\}$ c is arbitrary constant

14. The number are 1, 2, 3

15. Onion Rs. 5 /kg

Wheat Rs. 8/ kg

Rice Rs. 8/kg