LESSON 2

INVERSE TRIGONOMETRIC FUNCTIONS

1. DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

We know that $\tan\frac{\pi}{3} = \sqrt{3}$

This is written in inverse trigonometry as $\frac{\pi}{3} = \tan^{-1}\sqrt{3}$

But, $\tan\frac{4\pi}{3}$ is also equal to $\sqrt{3}$.

Does it mean : $\frac{4\pi}{3} = \tan^{-1}\sqrt{3}$?

The answer is no.

$\tan^{-1}\sqrt{3}$ is taken as the numerically least angle whose tangent is $\sqrt{3}$. This is done to associate a single value to $\tan^{-1}\sqrt{3}$ to safeguard the definition of a function.

So the equations $\tan x = y$ and $x = \tan^{-1} y$ are not identical because the former associates many values of x to a single value of y while the latter associates a single x to a particular value of y .In the same way, the remaining five inverse trigonometric functions are also defined .To assign a unique angle to a particular value of trigonometric ratio, we introduce a term called 'principal range'. The principal ranges of all the inverse trigonometric functions have been fixed. *e.g.,* Principal range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *i.e.,* We have to search for an angle in this interval only.

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ only, although $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\sin\frac{13\pi}{6} = \frac{1}{2}$, *etc*. $\left(\text{note that } \sin^{-1}\frac{1}{2} \neq \frac{1}{\sin\frac{1}{2}}\right)$.

Similarly, even if $\cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$ but $\cot^{-1}\left(-\sqrt{3}\right) \neq -\frac{\pi}{6}$ because principal range of $\cot^{-1}x$ is $(0, \pi)$.

so, $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$ only

The principal range of inverse trigonometric functions is the most important thing in this lesson. All formulae and problems are linked in some way or the other to that only.

We list below the domain and principal ranges of all the six inverse trigonometric functions.

Function	**Domain (values of *x*)**	**Principal Range (values of *y*)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

e.g., $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, $\text{cosec}^{-1}(1) = \frac{\pi}{2}$, $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3}$, *etc*.

Illustration 1

***Question*:** **Evaluate the following**

(a) $\tan^{J1}$ (J1), (b) $\cot^{J1}$ (J1), (c) $\sin^{J1}\left(-\frac{\sqrt{3}}{2}\right)$.

Solution : (a) $\tan\left(\frac{-\pi}{4}\right) = -1$, $\therefore \tan^{-1}(-1) = -\frac{\pi}{4}$ $\left\{\because \frac{-\pi}{4} \in \text{range of } \tan^{-1} x\right\}$

(b) $\cot\left(\frac{3\pi}{4}\right) = -1,$ $\quad\therefore\ \cot^{-1}(-1) = \frac{3\pi}{4}\ \left\{\because\ \frac{3\pi}{4} \in \text{range of } \cot^{-1} x\right\}$

(c) $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2},$ $\quad\therefore\ \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3}\ \left\{\because\ \frac{-\pi}{3} \in \text{range of } \sin^{-1} x\right\}$

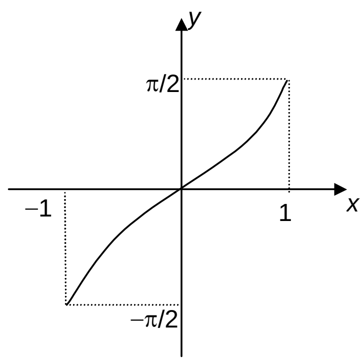
Illustration 2

***Question*:** **Simplify** $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) - \tan^{-1}\left(-\sqrt{3}\right) + \cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$**.**

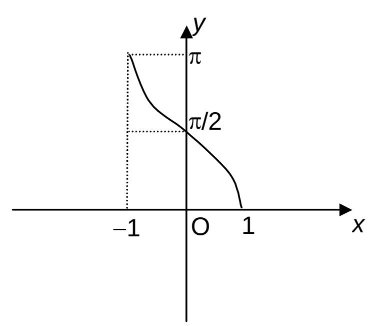
***Solution*:** The value $= \frac{-\pi}{4} + \frac{2\pi}{3} - \left(\frac{-\pi}{3}\right) + \left(\frac{2\pi}{3}\right)$

$$= -\frac{\pi}{4} + \frac{2\pi}{3} + \frac{\pi}{3} + \frac{2\pi}{3} = \frac{17\pi}{12}$$

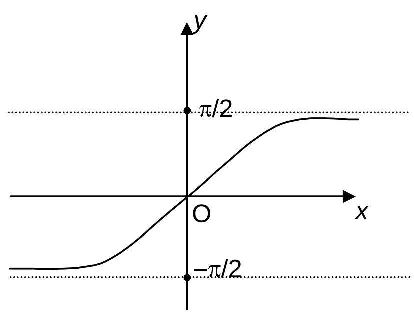
2. GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS



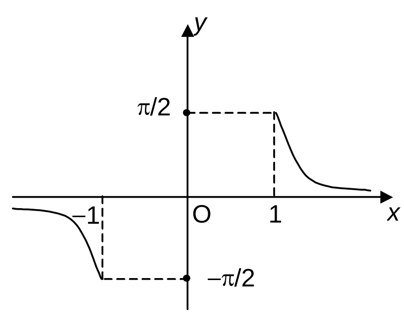
y* = sin^{-1}*x



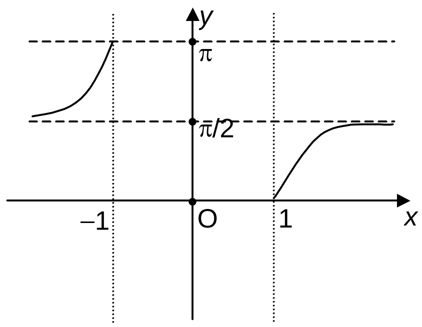
y* = cos^{-1}*x



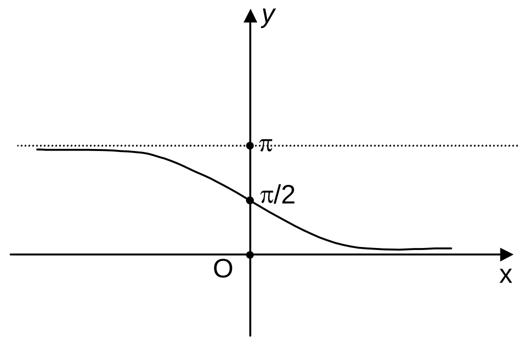
y* = tan^{-1}*x



y* = cosec^{-1}*x



$y = \sec^{J1}x$



$y = \cot^{J1}x$

3. SOME BASIC RESULTS

Observe the following : $\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \frac{7\pi}{6}$

In the first instance, it seems to be correct because there are two operations performed on $\frac{7\pi}{6}$ which are reverse of each other. But as we discussed, $\sin^{-1}x$ always lies in the range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, whereas $\frac{7\pi}{6}$ is clearly out of this range.

$$\therefore \qquad \sin^{-1}\left(\sin\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$$

$$\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

So, we have the following relations :

- $\sin^{-1}\sin\theta = \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- $\cos^{-1}\cos\theta = \theta$ if $0 \le \theta \le \pi$
- $\tan^{-1}\tan\theta = \theta$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ if $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$
- $\sec^{-1}(\sec\theta) = \theta$ if $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$
- $\cot^{-1}(\cot\theta) = \theta$ if $0 < \theta < \pi$

Similarly,

- $\sin(\sin^{-1}x) = x$ if $|x| \le 1$
- $\cos(\cos^{-1}x) = x$ if $|x| \le 1$
- $\tan(\tan^{-1}x) = x$ if $x \in R$

- $\text{cosec}(\text{cosec}^{-1} x) = x$ if $|x| \geq 1$
- $\sec(\sec^{-1} x) = x$ if $|x| \geq 1$
- $\cot(\cot^{-1} x) = x$ if $x \in R$

Illustration 3

***Question*:** **Evaluate each of the following:**

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ **(ii)** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(iii)** $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ **(iv)** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **(vi)** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution: Recall that $\sin^{-1}(\sin\theta) = \theta$, if $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, $\cos^{-1}(\cos\theta) = \theta$, if $0 \leq \theta \leq \pi$ and $\tan^{-1}(\tan\theta) = \theta$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore,

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

(ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

(iii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$ as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$ $\left[\because \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)\right]$

$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $[\because \sin(\pi - \theta) = \sin\theta]$

$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$

(v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$ $\left[\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right]$

$\Rightarrow \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$ $[\because \cos(2\pi - \theta) = \cos\theta]$

$\Rightarrow\ \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\frac{5\pi}{6}$

(vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)\neq\frac{3\pi}{4}$, because $\frac{3\pi}{4}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{4}\right)\right\}$ $\left[\because\ \frac{3\pi}{4}=\pi-\frac{\pi}{4}\right]$

$\Rightarrow\ \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$ $[\because\ \tan(\pi-\theta)=-\tan\theta]$

$\Rightarrow\ \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left\{\tan\left(-\frac{\pi}{4}\right)\right\}$ $\Rightarrow\ \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=-\frac{\pi}{4}$

Illustration 4

***Question*:** **Find the angle**

(a) $\mathbf{\tan^{J1}\left(\tan\frac{3®}{4}\right)}$;

(b) $\sin^{J1}$ sin 5 (where 5 is in radians).

***Solution*:** (a) Let $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\theta$

$\tan^{-1}\tan\left(\pi-\frac{\pi}{4}\right)=\theta$

$\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\theta$

$\Rightarrow\ -\tan^{-1}\tan\frac{\pi}{4}=\theta$ $\left[\text{As }\tan^{-1}\tan\theta=\theta,\text{if }\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow\ -\frac{\pi}{4}=\theta$

(b) We know $\sin^{-1}\sin\theta=\theta,\ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\approx[-1.57,\ 1.57]$ ...(i)

Hence $\sin^{-1}\sin 5\neq 5$ as $5\notin[-1.57, 1.57]$

$\therefore\ \sin 5=\sin\left(\pi+\overline{5-\pi}\right)$

$=-\sin(5-\pi)$

Since $(5-\pi)\notin[-1.57, 1.57]$ so we again add and subtract π.

$\Rightarrow\ \sin 5=-\sin\left(\pi+\overline{5-2\pi}\right)$

$=+\sin(5-2\pi)$ $[\because\ (5-2\pi)\in[-1.57, 1.57]]$

$\therefore \quad \sin^{-1} \sin 5 = \sin^{-1} \sin (5 - 2\pi) = 5 - 2\pi$

Hence to solve this type of problem, the procedure is to add and subtract π till it belongs to the principal value range of respective inverse trigonometric function.

4. FORMULAE IN INVERSE TRIGONOMETRY

We now discuss some important formulae in inverse trigonometry and their applications. It must be noted that the list is not exhaustive because all the formulae learnt in the first lesson of trigonometry can be converted to inverse trigonometry form, but there will be conditions attached with almost all of them due to the effect of principal range.

e.g., $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ only if $-1 \le x \le 1$

$$= \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad \text{if } x > 1$$

Let us see how

Suppose $\tan^{-1} x = \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow x = \tan \theta$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

$$= 2\theta \text{ only if } -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \text{ i.e., } -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

Since, $x = \tan\theta \Rightarrow -1 \le x \le 1$

If $x > 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$

So, we need to write

$\sin^{-1}(\sin 2\theta) = \sin^{-1}[\sin(\pi - 2\theta)]$

$$= \pi - 2\theta \quad \text{as } 0 < \pi - 2\theta < \frac{\pi}{2}$$

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x \quad \text{if } x > 1$$

Similarly, it can easily be shown that, $2\tan^{-1} x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ if $x < -1$

Consider few important results:

4.1 • $\sin^{-1}(-x) = -\sin^{-1}x \quad \forall\ x \in [-1, 1]$

Proof :

As $(-x) \in [-1, 1]$

$\therefore \quad x \in [-1, 1]$

Let $\sin^{-1}(-x) = \theta$, so $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad (-x) = \sin\theta$

$\Rightarrow \quad x = -\sin\theta$

$\Rightarrow \quad x = \sin(-\theta)$

$\Rightarrow \quad (-\theta) = \sin^{-1}x$

$\Rightarrow \quad \theta = -\sin^{-1}x$

$\sin^{-1}(-x) = -\sin^{-1}x, \ \forall x \in [-1, 1]$

• $\cos^{-1}(-x) = \pi - \cos^{-1}x \quad \forall\ x \in [-1, 1]$

Proof :

As $(-x) \in [-1, 1]$

$\therefore \quad x \in [-1, 1]$

Let $\cos^{-1}(-x) = \theta$, so $\theta \in [0, \pi]$

$\Rightarrow \quad -x = \cos\theta$

$\Rightarrow \quad x = \cos(\pi - \theta)$

$\Rightarrow \quad \cos^{-1}x = \pi - \theta$

$\Rightarrow \quad \theta = \pi - \cos^{-1}x$

$\Rightarrow \quad \cos^{-1}(-x) = \pi - \cos^{-1}x, \ \forall\ x \in [-1, 1]$

Try to derive the following formulae on your own.

• $\tan^{-1}(-x) = -\tan^{-1}x \quad \forall\ x \in R$

• $\cot^{-1}(-x) = \pi - \cot^{-1}x \quad \forall\ x \in R$

• $\sec^{-1}(-x) = \pi - \sec^{-1}x \quad \forall\ x \in (-\infty, -1] \cup [1, \infty)$

• $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x \quad \forall\ x \in (-\infty, -1] \cup [1, \infty)$

4.2 • $\tan^{-1}(1/x) = \begin{cases} \cot^{-1} x & \forall\ x > 0 \\ -\pi + \cot^{-1} x & \forall\ x < 0 \end{cases}$

Proof :

Let $\cot^{-1}x = \theta$

where, $-\infty < x < \infty$ and $0 < \theta < \pi$

Now, consider two cases,

Case I: $x > 0$

$\cot^{-1} x = \theta \quad \Rightarrow \quad \theta \in (0, \pi/2)$

$\Rightarrow \quad x = \cot\theta \quad \Rightarrow \quad \frac{1}{x} = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \tan^{-1}(1/x) = \cot^{-1}x$, for all $x > 0$.

Case II: $x < 0$

$\Rightarrow \quad \theta \in (\pi/2, \pi)$

$\pi/2 < \theta < \pi$

$\Rightarrow \quad -\pi/2 < (\theta - \pi) < 0$

$\Rightarrow \quad (\theta - \pi) \in (-\pi/2, 0)$

$\therefore \quad \cot^{-1}x = \theta$

$\Rightarrow \quad \cot\theta = x \quad \Rightarrow \quad \frac{1}{x} = \tan\theta$

$\Rightarrow \quad \frac{1}{x} = -\tan(\pi - \theta)$

$\Rightarrow \quad \frac{1}{x} = \tan(\theta - \pi)$

$\Rightarrow \quad (\theta - \pi) = \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1}x$, when $x < 0$.

$$\therefore \quad \tan^{-1}\frac{1}{x} = \begin{cases} \cot^{-1}x, & \forall \quad x > 0 \\ -\pi + \cot^{-1}x, & \forall \quad x < 0 \end{cases}$$

Similarly, within the domain of their definitions,

- $\tan^{-1}x = \begin{cases} \cot^{-1}\dfrac{1}{x} & , \; x > 0 \\ \cot^{-1}\left(\dfrac{1}{x}\right) - \pi & , \; x < 0 \end{cases}$

- $\sin^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1}x, \; \forall x \in (-\infty, -1] \cup [1, \infty)$

- $\sin^{-1}x = \operatorname{cosec}^{-1}\dfrac{1}{x}, \; \forall \; x \in [-1, 0) \cup (0, 1]$

- $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}x, \; \forall x \in (-\infty, -1] \cup [1, \infty)$

- $\cos^{-1}x = \sec^{-1}\dfrac{1}{x}, \; \forall \; x \in [-1, 0) \cup (0, 1]$

4.3 • $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2} \qquad , \quad -1 \le x \le 1$

• $\cot^{-1}x + \tan^{-1}x = \dfrac{\pi}{2} \qquad , \quad x \in R$

• $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2} \qquad , \quad -\infty < x \le -1$ or $1 \le x < \infty$

Proof:

• **$\sin^{-1}x + \cos^{-1}x = \pi/2, \; \forall \; x \in [-1, 1]$**

Let $\sin^{-1}x = \theta$, so $x \in [-1, 1]$

where, $\theta \in [-\pi/2, \pi/2]$

$\Rightarrow \quad -\pi/2 \le \theta \le \pi/2 \quad \Rightarrow \quad -\pi/2 \le -\theta \le \pi/2$

$\Rightarrow \quad 0 \le \pi/2 - \theta \le \pi \quad \Rightarrow \quad (\pi/2 - \theta) \in [0, \pi]$

$\because \quad \sin^{-1}x = \theta$

$\Rightarrow \quad x = \sin\theta$

$\Rightarrow \quad x = \cos(\pi/2 - \theta)$

$\Rightarrow \quad \cos^{-1}x = \pi/2 - \theta \quad \Rightarrow \quad \theta + \cos^{-1}x = \pi/2$

$\therefore \quad \sin^{-1}x + \cos^{-1}x = \pi/2$

Illustration 5

***Question*:** **Evaluate cos $[2\cos^{-1}x + \sin^{-1}x]$ at $x = \dfrac{1}{5}$.**

Solution: $2\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2} + \cos^{-1}x = \frac{\pi}{2} + \theta$ where $\cos\theta = x$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = -\sqrt{1-x^2} = -\sqrt{1-\frac{1}{25}} = \frac{-2\sqrt{6}}{5}$$

4.4 • $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, *if* $xy < 1, x > 0, y > 0$

Proof :

Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$. Then,

$x = \tan A$ and $y = \tan B$ and $A, B \in (-\pi/2, \pi/2)$.

$\therefore$ $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy}$

$x > 0$, $y > 0$ and $xy < 1$

$\Rightarrow$ $\frac{x+y}{1-xy} > 0$

$\tan(A+B) > 0$

$\Rightarrow$ $A + B$ lies in I quadrant or in III quadrant

$\Rightarrow$ $0 < A + B < \frac{\pi}{2}$ $\left[\begin{matrix}\because x > 0 \Rightarrow 0 < A < \frac{\pi}{2} \\ y > 0 \Rightarrow 0 < B < \pi/2\end{matrix}\right\} \Rightarrow 0 < A + B < \pi \Bigg]$

$\Rightarrow$ $\tan(A+B) = \frac{x+y}{1-xy}$

$\Rightarrow$ $A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $\left[\because 0 < A + B < \frac{\pi}{2}\right]$

$\Rightarrow$ $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

- $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$, **if** $xy \geq 0$, $x^2 + y^2 \leq 1$

  $= \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$, **if** $x > 0$, $y > 0$, $x^2 + y^2 > 1$

- $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$, **if** $xy > 0$, $x^2 + y^2 > 1$

- $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, **if** $-1 \leq x, y \leq 1$ **and** $x + y \geq 0$

  $= 2\pi - \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, **if** $-1 \leq x, y \leq 1$ **and** $x + y \leq 0$

- $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$, **if** $x \leq y$

  $= -\cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$, **if** $x \geq y$

Illustration 6

***Question*:** **Obtain the value of $\cos^{-1}\left(-\frac{3}{5}\right)+\sin^{-1}\left(-\frac{5}{13}\right)$ in terms of $\cos^{-1}$ function.**

Solution : $\cos^{-1}\left(-\frac{3}{5}\right)+\sin^{-1}\left(-\frac{5}{13}\right)$

$= \pi-\left(\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}\right)$

Let $\sin^{-1}\frac{4}{5}=\alpha \Rightarrow \sin\alpha=\frac{4}{5}$

$\sin^{-1}\frac{5}{13}=\beta \Rightarrow \sin\beta=\frac{5}{13}$

consider $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$= \frac{3}{5}.\frac{12}{13}-\frac{4}{5}.\frac{5}{13}=\frac{16}{65}$

$\Rightarrow \alpha+\beta=\cos^{-1}\frac{16}{65}$ ($\alpha, \beta \in$ quadrant 1)

$\therefore$ Given quantity = $\pi-\cos^{-1}\frac{16}{65}=\cos^{-1}\left(-\frac{16}{65}\right)$

Illustration 7

Question: **If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ prove that $x^2+y^2+z^2+2xyz=1$.**

Solution: Given: $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow$ $\cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z=\cos^{-1}(-z) \Rightarrow \cos[\cos^{-1}x+\cos^{-1}y]=\cos[\cos^{-1}(-z)]$

Let $\cos^{-1}x=A$

$\cos^{-1}y=B$

$\therefore$ $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\therefore$ $\cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$; $\therefore$ $(A+B)=\cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}]$

$\Rightarrow$ $\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}=\cos^{-1}(-z) \Rightarrow xy-\sqrt{1-x^2}\sqrt{1-y^2}=-z$

$\Rightarrow$ $(xy+z)^2=(1-x^2)(1-y^2) \Rightarrow x^2y^2+z^2+2xyz=1-x^2-y^2+x^2y^2$

$\Rightarrow$ $x^2+y^2+z^2+2xyz=1$.

Hence, proved.

Illustration 8

Question: **Write in the simplest form:**

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) \text{ where } -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

Solution:

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \frac{\pi}{4}-\frac{x}{2}.$$

PRACTICE PROBLEMS

PP1. Find the value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

PP2. Find the value of x for which $\tan^{-1} x = \frac{\pi}{2}$.

PP3. Find the value of $\sin(\cot^{-1}x)$ as an algebraic function of x.

PP4. Find the value of $\cos\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right)$.

PP5. Find the value of $\sec^{-1}[\sec(-30°)]$.

PP6. If $\cot^{-1}\alpha + \cot^{-1}\beta = \cot^{-1} x$, then find the value of x.

PP7. Find the value of $\tan^{-1}\tan\left(\frac{2\pi}{3}\right)$.

PP8. Find the value of $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{16}{65}$.

PP9. Find the value of $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$.

PP10. Find the value of $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}$.

SOLVED SUBJECTIVE EXAMPLES

Example* 1*:

If $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, find the value of *x.*

***Solution*:**

$\sin^{-1}x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$ ...(i)

Also, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ ...(ii)

Solving (i) and (ii), we get

$\sin^{-1}x = \frac{\pi}{3}$ and $\cos^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \frac{\sqrt{3}}{2}$ is the only solution.

Example 2:

Find the value of $\cos^{J1}\left(\cos\left(-\frac{®}{3}\right)\right)$.

Solution:

$\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}.$

Example 3:

Find the value of $\cot^{J1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$.

Solution:

Substituting $x^2 = \cos 2\theta$, we obtain

$\cot^{-1}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = \cot^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right) = \cot^{-1}(\tan\theta) = \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\theta\right)\right) = \frac{\pi}{2}-\theta.$

Example 4:

Find the value of $\sec^2 (\tan^{J1} 2) + \text{cosec}^2 (\cot^{J1} 3)$.

Solution:

$\sec^2 (\tan^{-1} 2) + \text{cosec}^2 (\cot^{-1} 3) = \{1 + \tan^2 (\tan^{-1} 2)\} + \{1 + \cot^2 (\cot^{-1} 3)\}$

$= 1 + \{\tan (\tan^{-1} 2)\}^2 + 1 + \{\cot (\cot^{-1} 3)\}^2 = 1 + 2^2 + 1 + 3^2 = 15$

Example 5:

If $\cos^{J1}\left(\frac{x}{a}\right) + \cos^{J1}\left(\frac{y}{b}\right) = \alpha$, prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$.

Solution:

Let $\cos^{-1}\left(\frac{x}{a}\right) = \theta; \cos^{-1}\left(\frac{y}{b}\right) = \phi$ so that $\theta + \phi = \alpha$

Now $\cos\alpha = \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi = \frac{x}{a}\cdot\frac{y}{b} - \sqrt{\left(1 - \frac{x^2}{a^2}\right)}\sqrt{\left(1 - \frac{y^2}{b^2}\right)}$

$\therefore \left(\frac{xy}{ab} - \cos\alpha\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\alpha = 1 - \cos^2\alpha = \sin^2\alpha$, which is the desired result.

Example 6:

Solve the equation $\tan^{-1}\left\{\frac{x+1}{x-1}\right\} + \tan^{-1}\left\{\frac{x-1}{x}\right\} = \tan^{-1}(-7)$.

Solution:

$$\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$$

taking tan on both sides, we get $\tan\left(\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right)\right) = \tan\tan^{-1}(-7)$

$$\therefore \quad \frac{\left(\frac{x+1}{x-1}\right) + \left(\frac{x-1}{x}\right)}{1 - \frac{x^2 - 1}{x(x-1)}} = -7$$

i.e., $\frac{2x^2 - x + 1}{1 - x} = -7 \Rightarrow 2x^2 - 8x + 8 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$

For $x = 2$, L.H.S. $= \tan^{-1}(3) + \tan^{-1}\left(\frac{1}{2}\right) > 0$ while R.H.S. $= \tan^{-1}(-7)$ is negative.

$\therefore$ no value of x satisfies the given equation.

Example 7:

Prove that $\sin\{\cot^{-1}(\tan(\cos^{-1}x))\} = |x|$.

Solution:

Let $\cos^{-1}x = \theta$; $\theta \in [0, \pi]$ $\Rightarrow$ $\cos\theta = x$, so $\tan\theta = \dfrac{\sqrt{1-x^2}}{x}$

Now let $\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \alpha,\ \alpha \in (0,\ \pi)$

$\therefore$ $\cot\alpha = \dfrac{\sqrt{1-x^2}}{x}$ $\Rightarrow$ $\sin\alpha = |x|$ (because $\sin\alpha$ is positive in $\alpha \in (0, \pi)$)

Example 8:

Solve $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x - 2)$.

Solution:

The given equation is $\dfrac{\pi}{2} - \cos^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$ or $\dfrac{\pi}{2} - 2\cos^{-1}x = \sin^{-1}(3x-2)$

$3x - 2 = \sin\left(\dfrac{\pi}{2} - 2\cos^{-1}x\right) = \cos(2\cos^{-1}x) = 2\cos^2(\cos^{-1}x) - 1 = 2x^2 - 1$

$\therefore$ $2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$\therefore$ $x = 1$ or $\dfrac{1}{2}$

Example 9:

Prove that $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$.

Solution:

We have, $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\left\{\dfrac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\} + \tan^{-1}\dfrac{1}{7}$ $\left[\because 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), \text{ if } -1 < x < 1\right]$

$= \tan^{-1}\dfrac{4}{3} + \tan^{-1}\dfrac{1}{7}$

$$= \tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\} = \tan^{-1}\frac{31}{71}$$

Example* 10*:

Prove that $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{x}{2},\ 0 < x < \frac{\pi}{2}$.

Solution:

We have, $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$

$$= \cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$\left[\because \left(\cos\frac{x}{2}\pm\sin\frac{x}{2}\right)^2 = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}\pm 2\sin\frac{x}{2}\cos\frac{x}{2} = 1\pm\sin x\right]$$

$$= \cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\} \qquad [\because \sqrt{x^2} = |x|]$$

$$= \cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \qquad \left[\because 0<\frac{\pi}{2}<\frac{\pi}{4}, \therefore \cos\frac{x}{2} > \sin\frac{x}{2}\right]$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} \qquad \left[\because 0<\frac{x}{2}<\frac{\pi}{4}\right]$$

EXERCISE – I

1. Find the principal values of the following:

(i) $\sin^{-1}\left(-\frac{1}{2}\right)$ (ii) $\tan^{-1}\left(-\sqrt{3}\right)$ (iii) $\sec^{-1}(-2)$

2. Evaluate the following:

(i) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$.

(ii) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$.

3. Find the value of $\sin\left(\sin^{-1}x+\cos^{-1}x\right)$.

4. Find the values of each of the following:

(i) $\tan^{-1}\left(\tan\frac{7\pi}{4}\right)$ (ii) $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

5. Prove that $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

6. Prove that $\tan^{-1}\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}=\frac{\theta}{2},\ \theta\in(0,\ \pi)$.

7. Solve the equations for x and y

$$\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$$

$$\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}.$$

8. Find the value of $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$, when

(i) $0\le x\le 1$ (ii) $x>1$

9. Simplify: $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$.

10. Solve for x if $\sin^{-1}\left(\frac{2a}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

11. Find the value of $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$.

12. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, prove that $x + y + z = xyz$.

13. Solve: $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$, $0 < x < 1$

14. Evaluate sin $\cot^{-1}$ cos ($\tan^{-1} x$) as an algebraic function of *x.*

15. Show that: $\sin^{-1}$ (sin 3) + $\cos^{-1}$ (cos 7) – $\tan^{-1}$ (tan 5) = $\pi - 1$

EXERCISE – II

1. Find the value of $\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)$.

2. Find the value of $\tan\left(2\cos^{-1}\frac{3}{5}\right)$.

3. Find the value of $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]$.

4. Find the value of $\cos^{-1}(3/5)+\cos^{-1}(4/5)$.

5. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then find the value of $(xy+yz+zx)$.

6. If $6\sin^{-1}(x^2-6x+8.5)=\pi,$ find the value of x.

7. Solve the equation $\tan^{-1}\left(\tan\sqrt{1-\theta}\right)=\sqrt{1-\theta}$.

8. If $\sin^{-1}x=\pi/5,$ for some $x\in[-1,1],$ then find the value of $\cos^{-1}x$.

9. Find the values of x for which $\tan(\sec^{-1}x)=\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$.

10. Solve $\tan^{-1}\left(\frac{x-1}{x+1}\right)+\tan^{-1}\left(\frac{2x-1}{2x+1}\right)=\tan^{-1}\frac{23}{36}$.

11. Find the value of $\tan\left[\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right]$.

12. If $\cot^{-1}x=2\cot^{-1}7+\cos^{-1}(3/5),$ then find the value of x.

13. Find the set of values of x satisfying the inequation $\tan^2(\sin^{-1}x)>1$.

14. Find the value of $\cos^{-1}\left[\cos\left(-\frac{17}{15}\pi\right)\right]$.

15. Find the smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right), 0\le x\le 1$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\frac{\pi}{3}$

PP2. does not exist

PP3. $\frac{1}{\sqrt{1+x^2}}$

PP4. $\frac{1}{\sqrt{2}}$

PP5. $\frac{\pi}{6}$

PP6. $\frac{\alpha\beta-1}{\alpha+\beta}$

PP7. $-\frac{\pi}{3}$

PP8. $\frac{\pi}{2}$

PP9. $\cos^{-1}\frac{36}{85}$

PP10. $\cos^{-1}\frac{33}{65}$

EXERCISE – I

1. (i) $-\frac{\pi}{6}$ (ii) $-\frac{\pi}{3}$ (iii) $\frac{2\pi}{3}$

2. (i) $\frac{1}{2}$

(ii) 1

3. 1

4. (i) $-\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$

7. $x = \frac{1}{2},\ y = 1$

8. (i) $\frac{2x}{1-x^2}$ (ii) not finite

9. $\tan^{-1}\frac{a}{b} - x$

10. $\frac{a-b}{(1+ab)}$

11. $\frac{\pi}{4}$

13. $x = \frac{1}{4}$

EXERCISE – II

1. 1

2. $-\frac{24}{7}$

3. $-\frac{7}{17}$

4. $\pi/2$

5. 3

6. $x = 2$

7. $\frac{4-\pi^2}{4} < \theta \le 1$

8. $\frac{3\pi}{10}$

9. $\pm\frac{3}{\sqrt{5}}$

10. $x = \frac{4}{3}$

11. $\frac{3}{29}$

12. 44/117

13. $(-1,1)-\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

14. $\sqrt{\frac{1+x^2}{2+x^2}}$

15. $0, \pi/4$