LESSON 1

RELATIONS AND FUNCTIONS

1. RELATIONS

1.1 INTRODUCTION

In our day to day life, we often talk about relation between two persons, between two straight lines (e.g. perpendicular lines, parallel lines) etc.

Let A be the set of all male students in Delhi whose fathers live in Delhi. Let B be the set of all the people living in Delhi. Let a be a male student living in Delhi i.e. $a \in A$. Let b be the father of a. Then $b \in B$. And a is related to b under son-father relation. If we denote the son-father relation by symbol R then a is related to b under relation R. We can also express this by writing aRb. Here R denotes the relation 'is son of'.

We can also express this statement by saying that the pair of a and b is in relation R i.e., the ordered pair $(a, b) \in R$. This pair (a, b) is ordered in the sense that a and b can't be interchanged because first co-ordinate a represents son, and the second coordinate b represents father of a. Similarly if $a_1 \in A$ and b_1 is father of a_1, then $(a_1, b_1) \in R$. So we can think of the relation R as a set of ordered pairs whose first coordinate is in A and the second coordinate is in B. Thus $R \subseteq A \times B$. Since the relation 'is son of' i.e., R is a relation relating elements of A to be elements of B, we will say that R is a relation from set A to set B.

Hence a relation R from non empty set A to a non empty set B is a subset of the Cartesian product $A \times B$. This subset is derived by describing a relationship between first element and second element of the ordered pairs in $A \times B$.

DEFINITION 1

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$

DEFINITION 2

A relation R in a set A is called universal relation, if each element of A is related to any element of A, i.e., $R = A \times A$

Both, the empty relation and the universal relation are sometimes called trivial relations.

DEFINITION 3

A relation R in a set A (non empty set) is called

(i) Reflexive, if $(a, a) \in R$, for every $a \in A$

(ii) Symmetric, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$

(iii) Transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

DEFINITION 4

A relation R in a set A (non empty) is said to be an equivalence relation, if R is reflexive, symmetric and transitive.

Definition of equivalence class:

Let A be a nonempty set and R be an equivalence relation on A. Let a be an element of A.

The set of all elements of A which are R related to a is denoted by [a] and is called an equivalence class containing a.

Thus $[a] = \{x : x \in R,\ x R a \ \ i.e.,\ (x, a) \in R\}$

It is noteworthy that an equivalence relation R on a set A divides A into nonempty mutually disjoint subsets A_i, called partitions of A satisfying the following conditions:

(i) All elements of A_i are related to each other for all i.

(ii) For $i \neq i$, no elements of A_i is related to any element of A_i.

(iii) $A = \cup A_i$

The subsets A_i are called equivalence classes.

The converse of the above statement is also true i.e., corresponding to every partition A_1, A_2........ of a set A, there exists an equivalence relation R on set A for which these subsets are distinct equivalence classes. In fact this equivalence relation R defined as $(a, b) \in R$ i.e., $a R b \Leftrightarrow$ a and b belong to the same subset A_i

***Illustration* 1**

Question: **Show that the relation *R* in the set *Z* of integers given by**

$R = \{(a, b) : 2$ divides $a - b\}$

is an equivalence relation.

***Solution*:** R is reflexive, as 2 divides ($a - a$) for all $a \in Z$.

Further, if $(a, b) \in R$, then 2 divides $a - b$.

Therefore, 2 divides $b - a$.

Hence $(b, a) \in R$, which shows that R is symmetric.

Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2.

Now, $a - c = (a - b) + (b - c)$ is even (why?).

So, $(a - c)$ is divisible by 2.

This shows that R is transitive. Thus, R is an equivalence relation in Z.

2. DOMAIN AND RANGE OF A RELATION

2.1 DOMAIN OF A RELATION

Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in R$. Domain of R is precisely written as domain R.

Thus domain of (R) = $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$

Thus domain of R = set of first components of all the ordered pair which belong to R.

2.2 RANGE OF A RELATION

Let R be a relation from A to B. The range of R is the set of all those elements $b \in R$ such that $(a, b) \in R$ for some $a \in A$.

Thus range of $R = \{b \in B : (a, b) \in R \text{ for some a} \in A\}$.

Range of R = set of second components of all the ordered pairs which belong to R.

Set B is called as codomain of relation R.

Example1: Let $A = \{2, 3, 5\}$ and $B = \{4, 7, 10, 8\}$

Let $aRb \Leftrightarrow a$ divides b

Then $R = (2, 5)$ and range of $R = \{4, 10, 8\}$

Codomain of $R = B = \{4, 7, 10, 8\}$

2.3 Number of Relations

Let A and B be two non empty finite sets having p and q elements respectively.

Then $n(A \times B) = n(A).n(B) = pq$

Therefore, total number of subsets of $A \times B = 2^{pq}$

3. DEFINITION OF FUNCTION

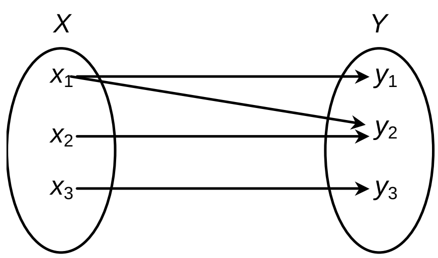
Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f ' then mathematically we write $f : X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under 'f ' (or x is the pre image of y).

So we can say that for a function, following two conditions must be satisfied:

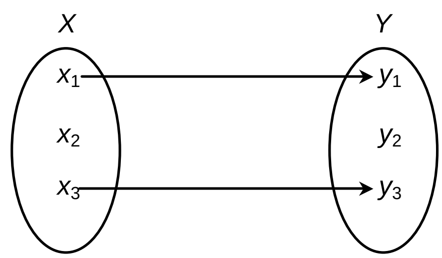
(i) *A* mapping $f : X \to Y$ is said to be *a* function if each element in the set X has it's image in set Y. It is possible that *a* few elements in the set Y are present which are not the images of any element in set X.

(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for *a* specific element in set X. Functions can't be multi-valued.

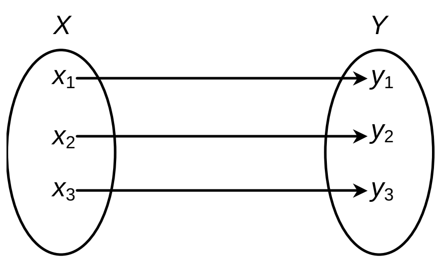
For $y = f(x)$ (y is *a* function of x), x is called independent variable and y is called dependent variable.



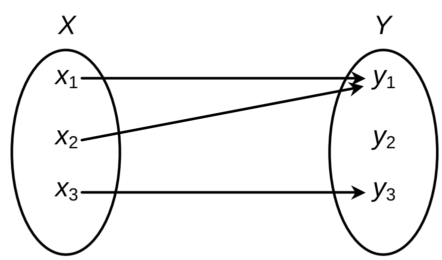
Not a function



Not a function



Function

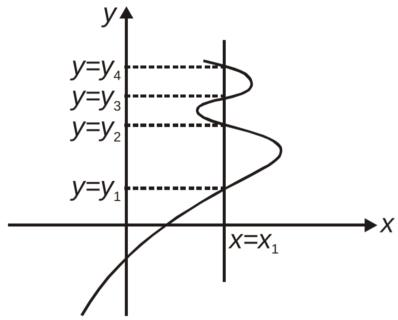


Function

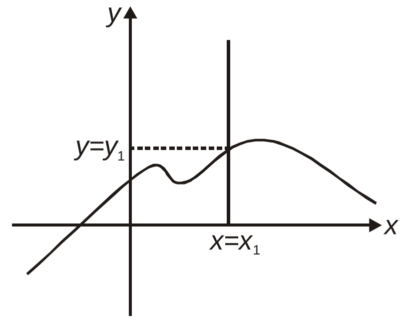
Let us take an example of $y^2 = x$. Here y cannot be *a* function of x as $y = \pm\sqrt{x}$ and for each value of $x \in \boldsymbol{R}$, we get two values of $y \in \boldsymbol{R}$, which is in contradiction to the above specified conditions, for a relation to be a function. So $y = \pm\sqrt{x}$ is not *a* function but is a relation.

Let us take another example of $f : \boldsymbol{R} \to \boldsymbol{R}$, $y = x^3$. Here for each and every value of $x \in \boldsymbol{R}$, we have unique value of $y \in \boldsymbol{R}$. As both the conditions are satisfied, $y = x^3$ is said to be a function of x.

Graphically, if *a* line parallel to y-axis cuts the graph of the relation between x and y, in not more then one point, the relation is said to be a function in x or in other words, y is said to be a function of x.



y* is not a function of *x



y* is a function of *x

Illustration 2

Question: **If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.**

***Solution*:** We have, $f(x) = x + \frac{1}{x} = f\left(\frac{1}{x}\right)$

$\therefore \quad f(x^3) = x^3 + \frac{1}{x^3}$

and $\quad [f(x)]^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = f(x^3) + 3f(x) = f(x^3) + 3f\left(\frac{1}{x}\right)$

4. DOMAIN AND CODOMAIN OF FUNCTION

Let $f : X \to Y$ be *a* function. In general, sets *X* and *Y* could be any arbitrary non-empty sets. But at this level we would confine ourself only to real valued functions. That means it would be invariably assumed that *X* and *Y* are the subsets of real numbers.

Set '*X* ' is called domain of the function '*f* '.

Set '*Y* ' is called the co-domain of the function '*f* ',

In some cases, when domain of *a* function is not explicitly defined, domain would mean the set of real values of *x*, for which *f*(*x*) assumes real values. In other words, all the values of $x \in \boldsymbol{R}$ for which *f*(*x*) is non real or not defined, will not be an element of the domain. Domain of *a* function '*f* ' is normally represented as domain (*f*).

For a function $y = f(x) = \frac{1}{2x+1}$

Domain (*f*) = {*x* : *f*(*x*) is a real number}

Here *f*(*x*) is real for each and every value of $x \in \boldsymbol{R}$ except $x = -\frac{1}{2}$ (the value of *x* at which the denominator is becoming zero). So domain of the above function is defined as $x \in \boldsymbol{R} - \left\{-\frac{1}{2}\right\}$.

Let us take another example of $f(x) = \sqrt{(x-1)(x-2)}$.

Here for $f(x)$ to be real, $(x - 1)(x - 2) \geq 0$

$\Rightarrow \quad x \in (-\infty, 1] \cup [2, \infty)$.

So domain of the function would be $x \in (-\infty, 1] \cup [2, \infty)$, as for only these values of x, $f(x)$ assumes real values.

Illustration 3

Question: **Find the domain of the function $f(x) = \sqrt{x^2 - 3x + 2}$.**

Solution: Let $y = f(x) = \sqrt{x^2 - 3x + 2}$

For D_f, $x^2 - 3x + 2 \geq 0 \;\Rightarrow\; (x-1)(x-2) \geq 0$

Two cases arise: either $x - 1 \geq 0$

and $x - 2 \geq 0$ or $x - 1 \leq 0$ and $x - 2 \leq 0$

$\Rightarrow \; x \geq 1$ and $x \geq 2$ or $x \leq 1$ and $x \leq 2$

$\Rightarrow \; x \geq 2 \;\Rightarrow\; x \leq 1$

Hence, the domain of $f(x)$ is the set of those real which are less than or equal to 1 or which are greater than or equal to 2 i.e., $D_f\,(-\infty, 1] \cup [2, \infty)$.

5. RANGE OF FUNCTION

Set of images of all elements of set X is called the range of the function.

i.e. range is a set consisting of all the values of $f(x)$ as its elements, which we get for each and every value of $x \in$ domain.

It is obvious that range could be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of the set X (of course these elements of co-domain will not be included in the range). Range of *a* function is normally represented as Range (f).

There is no specific method to find out range of *a* function, as the type of method varies with different types of functions. Still to make the idea clear some methods are discussed as illustrations below according to different functions.

Illustration 4

Question: **Find the range of the function $f(x) = \dfrac{1}{8 - 3\sin x}$.**

Solution: $f(x) = \dfrac{1}{8 - 3\sin x}$

$-1 \le \sin x \le 1$

$\Rightarrow \quad -3 \le 3\sin x \le 3 \quad \Rightarrow \quad 5 \le 8 - 3\sin x \le 11 \quad \Rightarrow \quad \frac{1}{11} \le \frac{1}{8-3\sin x} \le \frac{1}{5}$

$\therefore \quad \text{Range } (f) = \left[\frac{1}{11}, \frac{1}{5}\right]$

6. CLASSIFICATION OF FUNCTIONS

6.1 ONE-ONE AND MANY-ONE FUNCTIONS

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one functions are also called injective function.

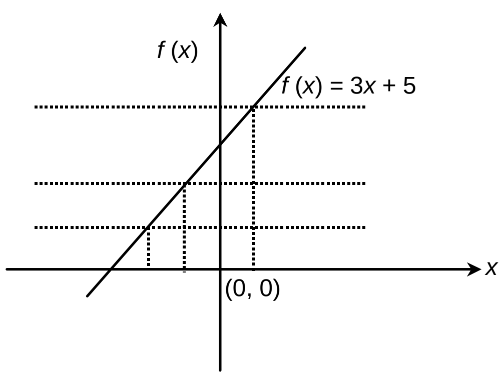
e.g. $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 3x + 5$ is one-one.

On the other hand, if there are at least two elements in the domain whose images are the same, the function is known as many-one.

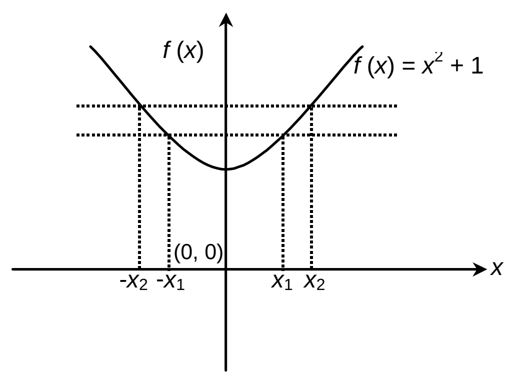
e.g. $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2 + 1$ is many-one.

Note that a function will either be one-one or many-one.

You might be inquisitive about methods of determining one-one and many-one functions. So, if you have understood the very distinction between these two types of function, you will agree with us that lines drawn parallel to the x-axis from the each corresponding image point shall intersect the graph of $y = f(x)$ at one (and only one) point if $f(x)$ is one-one and there will be at least one line parallel to x-axis that will cut the graph at least at two different points, if $f(x)$ is many-one and vice versa.



Graph of $f(x) = 3x + 5$



Graph of $f(x) = x^2 + 1$

Note that, a many-one function can be made one-one by redefining the domain of the original function.

Methods to determine one-one and many-one:

- Let x_1 , $x_2 \in$ domain of f and if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- Conversely if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.
- Any continuous function $f(x)$, which has at least one local maxima or local minima is many-one.
- All even functions are many one.

- All polynomials of even degree defined in R have at least one local maxima or minima and hence are many one in the domain R. Polynomials of odd degree can be one-one or many-one.
- If f is a rational function then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence we can write $f(x_1) - f(x_2) = (x_1 - x_2)\, g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-one else one-one.
- Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-one or many-one.

Illustration 5

Question: **Show that the function $f(x) = \frac{x^2 - 8x + 18}{x^2 + 4x + 30}$ is not one-one.**

***Solution*:** A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (only)

Now $f(x_1) = f(x_2) \Rightarrow \frac{x_1^2 - 8x_1 + 18}{x_1^2 + 4x_1 + 30} = \frac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$

$$\Rightarrow 12x_1^2 x_2 - 12x_1 x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$$

$$\Rightarrow (x_1 - x_2)\{12x_1 x_2 + 12(x_1 + x_2) - 312\} = 0$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 = \frac{26 - x_2}{1 + x_2}$$

Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ alone, $f(x)$ is not a one-one function.

6.2 ONTO AND INTO FUNCTIONS

We have another classification called onto or into functions. Let $f : X \to Y$ be a function. If each element in the codomain 'Y' has at least one pre-image in the domain 'X' that is, for every $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$, then f is onto. In other words range of $f = Y$, for onto functions.

On the other hand, if there exists at least one corresponding element in the codomain 'Y' which is not an image of any element in the domain X, then f is into.

Onto function is also called surjective function.

e.g. $f : \boldsymbol{R} \to \boldsymbol{R}$ where $f(x) = ax^3 + b$ is onto where $a \neq 0$, $b \in \boldsymbol{R}$.

Note that a function will either be onto or into.

Methods to determine whether a function is onto or into :

- If range = codomain, then f is onto. If range is a proper subset of codomain, then f is into.
- Solve $f(x) = y$ for x, say $x = g(y)$.

Now, if $g(y)$ is defined for each $y \in$ codomain and $g(y) \in$ domain of f for all $y \in$ codomain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in codomain, then $f(x)$ is into.

Note: Any polynomial function $f : \boldsymbol{R} \to \boldsymbol{R}$ is onto, if degree is odd; into, if degree of f is even.

Illustration 6

Question: **Let *f* : *N* ì *I* be a function defined as *f*(*x*) = *x* J 1000. Show that *f* is an into function.**

***Solution*:** Let $f(x) = y = x - 1000$

$\Rightarrow \quad x = y + 1000 = g(y)$ (say)

Here $g(y)$ is defined for each $y \in \boldsymbol{I}$, but $g(y) \notin N$ for $y \le -1000$.

Hence f is into.

6.3 BIJECTIVE FUNCTION

The function which is one-one and onto both is called bijective function.

Illustration 7

Question: **Show that *f : R* ì *R, f*(*x*) Z $\frac{x}{1\text{H}|x|}$ is not bijective.**

***Solution*:** Let x_1 and x_2 be both positive.

$$f(x_1)=f(x_2)\Rightarrow \frac{x_1}{1+|x_1|}=\frac{x_2}{1+|x_2|}\Rightarrow\frac{x_1}{1+x_1}=\frac{x_2}{1+x_2}$$
$$\Rightarrow x_1=x_2$$

If x_1 and x_2 are both negative, then $f(x_1)=f(x_2)\Rightarrow \frac{x_1}{1-x_1}=\frac{x_2}{1-x_2} \Rightarrow x_1=x_2$.

If x_1 and x_2 are of opposite sign, then $\frac{x_1}{1+|x_1|}=\frac{x_2}{1+|x_2|}$ cannot be true since the denominators are positive. i.e. if x_1 and x_2 are of opposite sign, then they cannot have the same image.

Hence the function is one-one

But if $y = \frac{x}{1+|x|}$, then y is numerically less than 1.

Hence the function cannot take values greater than or equal to 1.

Therefore the function is not onto, hence not bijective.

7. COMPOSITE FUNCTION

Let f and g be two functions with domain D_1 and D_2 respectively. If range $(f) \subset$ domain (g), we define gof by the rule

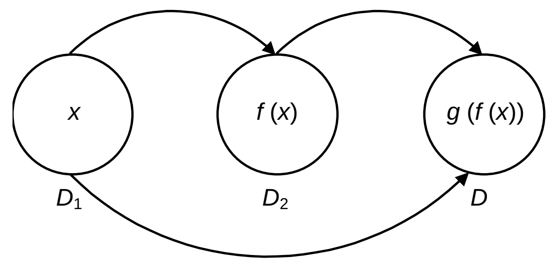
$$(gof)(x) = g(f(x)) \text{ for all } x \in D_1.$$

Also, if range (g) $\subset$ domain (f), we define fog by the rule

$(fog)(x) = f(g(x))$ for all $x \in D_2$.

Let us say $h(x) = gof(x)$

To obtain $h(x)$, we first take the f-image of an element $x \in D_1$ so that $f(x) \in D_2$, which is the domain of $g(x)$. Then take g-image of $f(x)$, i.e. $g(f(x))$ which would be an element of D. The figure given below clearly shows the steps to be taken. The function h defined above is called the composition of f and g and is denoted by gof. Thus $(gof)\,x = g(f(x))$.



Clearly domain (gof) = $\{x : x \in \text{domain}(f), f(x) \in \text{domain}(g)\}$

Similarly we can define, $(fog)x = f(g(x))$ and domain (fog) = $\{x : x \in \text{domain}(g), g(x) \in \text{domain}(f)\}$. In general $fog \neq gof$.

Illustration 8

Question: **Let f be the sine function and let g be the function $2x$. Find**

(i) *fog* (ii) *gof* (iii) *fof* (iv) *gog*

***Solution*:** We have $f(x) = \sin x$ and $g(x) = 2x$

(i) $(fog)(x) = f(g(x)) = f(2x) = \sin 2x$

(ii) $(gof)(x) = g(f(x)) = g(\sin x) = 2\sin x$

(iii) $(fof)(x) = f(f(x)) = f(\sin x) = \sin(\sin x)$

(iv) $(gog)(x) = g(g(x)) = g(2x) = 2(2x) = 4x$

8. INVERSE OF A FUNCTION

If $f : X \to Y$ be a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also if g is the inverse of f, then f is the inverse of g and the two functions f and g are said to be inverses of each other.

The condition for existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

Note that $fof^{-1}(x) = f[f^{-1}(x)] = x$ always, it means f and f^{-1} are symmetric about the line $y = x$.

Methods of finding the inverse of a function

- If you are asked to check whether the given function $y = f(x)$ is invertible, you need to verify that $y = f(x)$ is one-one and onto.
- If you are asked to find the inverse of a bijective function $f(x)$, you do the following:

  If f^{-1} be the inverse of f, then

  $f[f^{-1}(x)] = fof^{-1}(x) = x$ (always)

  Apply the formula of f on $f^{-1}(x)$ and use the above identity to solve for $f^{-1}(x)$.

Illustration 9

Question: **Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to$ range (f) is invertible. Find the inverse of f.**

***Solution*:** In order to prove that f is invertible, it is sufficient to show that $f : N \to$ range (f) is a bijection.

f is one-one: For any $x, y \in N$, we find that

$f(x) = f(y)$

$\Rightarrow \quad 4x^2 + 12x + 15 = 4y^2 + 12y + 15$

$\Rightarrow \quad 4(x^2 - y^2) + 12(x - y) = 0 \quad \Rightarrow \quad (x - y)(4x + 4y + 3) = 0$

$\Rightarrow \quad x - y = 0 \qquad [\because\ 4x + 4y + 3 \neq 0$ for any $x, y \in N]$

$\Rightarrow \quad x = y$

So, $f : N \to$ Range (f) is one-one.

Obviously, $f : N \to$ Range (f) is onto.

Hence, $f : N \to$ Range (f) is invertible.

Let f^{-1} denote the inverse of f. Then $fof^{-1}(x) = x$ for all $x \in$ range (f)

$\Rightarrow \quad f(f^{-1}(x)) = x$ for all $x \in$ range (f)

$\Rightarrow \quad 4\{f^{-1}(x)\}^2 + 12f^{-1}(x) + 15 = x$ for all $x \in$ range (f)

$$\Rightarrow \quad f^{-1}(x) = \frac{-12 \pm \sqrt{144 - 16(15 - x)}}{8} = \frac{-3 \pm \sqrt{x - 6}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{-3 + \sqrt{x - 6}}{2} \qquad [\because\ f^{-1}(x) \in N \ \therefore\ f^{-1}(x) > 0]$$

9. BINARY OPERATION

An operation is a process which produces a new element from two given elements; e.g., addition, subtraction, multiplication and division of numbers. If the new element belongs to the same set to which the two given elements belong, the operation is called a binary operation.

We know the operation of addition and multiplication of numbers, the operation of union and intersection of sets and the operation of composition of two functions. In all these operations, when two elements are operated, a new element is formed.

DEFINITION

A binary operation (or binary composition) denoted by * or $\odot$ on a non-empty set *A* is a mapping which associates with each ordered pair (*a*, *b*) $\in A \times A$ a unique element (to be denoted by *a***b*) of *A*.

Thus a binary operation * on *A* is a mapping *: $A \times A \to A$ defined by * (*a*, *b*) of *A*.

Example : Addition (+) as well as multiplication (×) are binary operations on the sets *N*, *Z*, *Q*, *R* and *C*.

$a, b \in N \Rightarrow a * b \in N$

i.e. if $a, b \in N \Rightarrow a \times b \in N$

Multiplication (×) is not a binary operation on the set of irrationals.

For, $\sqrt{2} \cdot \sqrt{2} = 2 \notin Y$ where as $\sqrt{2} \in Y$.

Similarly, subtraction (–) is not a binary operation on *N*. Also division is not a binary operation on *Z*. The division (÷) on R_0 i.e. *R* – {0} can be defined as ÷ : $R_0 * R_0 \to R_0$ by the rule ÷ (*a*, *b*) = *a* ÷ *b*.

Clearly ÷ is a binary operation on R_0.

NUMBER OF BINARY OPERATIONS ON A FINITE SET

Let *A* be a finite set containing *m* elements, then number of elements in the Cartesian product $A \times A = m \times m = m^2$. We know that if the set *A* has *m* elements and the set *B* has *n* elements, then number of functions from *A* to $B = n^m$.

Since a binary operations on set *A* is a function from $A \times A \to A$, therefore number of binary operations on *A.*

= number of functions from *A* × *A* to *A* = $[n(A)]^{n(A\times A)} = m^{m^2}$.

Thus if the set *A* has *m* elements, then number of binary operations on *A* is $(m)^{m^2}$.

Example : Let *A* = {*a*, *b*}, then $n(A) = 2$.

$\therefore$ Number of binary operations on $A = (2)^{2^2} = 2^4 = 16$.

9. TYPES OF BINARY OPERATIONS

- **Commutative Binary Operation :** Consider a binary operation '*' on a set *S*. For any two distinct elements in *S*, *a* * *b* and *b* * *a* may or may not be equal. Thus, it is not necessary that for a binary operation * on a set *S*, *a* * *b* = *b* * *a* for all *a*, *b* $\in$ *S*. If *a* * *b* = *b* * *a* for *a*, *b* $\in$ *S*, then we say that the binary operation * is commutative.

Example: The binary operations addition (+) and multiplication (×) are commutative binary operations on *Z*. However, the binary operation subtraction (–) is not a commutative binary operation on *Z* as $2 - 1 \neq 1 - 2$.

If the binary operation * on set *S* is commutative, we say that the system (*S*, *) is commutative.

- **Associative Binary Operation :** A binary operation '*' on a set *S* is said to be an associative binary operation, if

  $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

  If the binary operation * on set *S* is associative, we also say that the system (*S*, *) is associative.

Illustration 10

Question: **The binary operations of addition (+) and multiplication (×) are associative binary operations on *Z*. However, the binary operation subtraction (–) is not an associative binary operation on Z as (2 J 3) J 5 ÷ 2 J (3 J 5).**

If *S* is a non-empty set, then union (Ć) and intersection (ą) are both commutative and associative binary operations on *P*(*S*) (power set of *S*).

***Solution*:** $A \cup B = B \cup A,\ A \cap B = B \cap A$

$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ for all $A, B, C \in P(S)$.

- **Distributive Binary Operation:** Let *S* be a non-empty set and * and '$\odot$', be two binary operations on *S*. Then, '*' is said to be distributive over $\odot$, if

  $a * (b \odot c) = (a * b) \odot (a * c)$ and $(b \odot c) * a = (b * a) \odot (c * a)$ for all $a, b, c \in S$.

Example: The binary operation multiplication (·) on *Z* is distributive over the binary operation addition (+) on *Z*, because

$a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for al $a, b, c \in Z$.

However, addition (+) is not distributive over multiplication (·), because

$$2 + (3 \times 5) \neq (2 + 3) \times (2 + 5).$$

If *S* is a non-empty set, then union ($\cup$) is distributive over intersection ($\cap$) on $P(S)$, for

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \text{ for all } A, B, C \in P(S).$$

Clearly, intersection ($\cap$) is distributive over union ($\cup$) on *P*(*S*).

10. IDENTITY ELEMENT

Let * be a binary operation on a non empty set *A*. An element $e \in A$ is called an identity element for the binary operation * if $e * a = a * e = a$ for all $a \in A$.

Example: Let R be the set of all real numbers and * be the operation of ordinary multiplication (×) of integers.

Then $1 \in R$ is the identity element for the operation '×' because $1 \times a = a \times 1 = 1$ for all $a \in R$.

THEOREM : Identity element for a binary operation if it exists is unique.

Proof: Let * be a binary operation on a non empty set A.

Let $e \in A$ be an identity element for the binary operation *.

If possible, let there be another identity element $e_1 \in A$ for the binary operation *.

Now since e is the identity element for *

$\therefore \quad e * a = a * e = a$ for every $a \in A$

$\Rightarrow \quad e * e_1 = e_1 * e = e_1$ [putting e_1 in place of a]

$\Rightarrow \quad e * e_1 = e_1$ …(i)

Again e_1 is the identity element for *

$\therefore \quad e_1 * a = a * e_1 = a$ for every $a \in A$

$\Rightarrow \quad e_1 * e = e * e_1 = e$ [putting e in place of a]

$\Rightarrow \quad e_1 * e = e * e_1 = e$ …(ii)

From (i) and (ii), we have $e_1 = e$.

Hence identity is unique.

10. INVERSE OF AN ELEMENT AND INVERTIBLE ELEMENT

Let * be a binary operation on a non empty set A and e be the identity element for the binary operation *. Then an element $a \in A$ is said to be **invertible** with respect to binary operation * if there exists an element $b \in A$ such that $a * b = b * a = e$. The element is called an **inverse** of element a and is denoted by a^{-1}

Thus an element $a \in A$ is invertible if and only if its inverse exists.

Example: Let Z be the set of all integers and * be the operation of ordinary addition.

Then $0 \in Z$ is the identity element.

Let a be any element of Z. Then inverse of a for operation * is $-a \in Z$ because

$a + (-a) = (-a) + a = 0$ (identity element).

Example: Let Z be the set of all integers and * be the operation of ordinary multiplication.

Then only $1 \in Z$ is invertible.

For example, $2 \in Z$ has no inverse in Z as if $b \in Z$, such that $2 \times b = b \times 2 = 1$

Then $b = \frac{1}{2} \notin Z$.

In fact no element $a \in Z$, $a \div 1$ has inverse in Z.

THEOREM 1. Let '*' be an associative binary operation on a set S with the identity element e in S. Then, the inverse of an invertible element is unique.

Proof :

Let a be an invertible element in S.

If possible, let b and c be two inverse of $a \in S$ with respect to '*'.

Then, $a * b = b * a = e$ …(i)

And $a * c = c * a = e$ …(ii)

Now, $(b * a) * c = e * c$ [$\because$ $b * a = e$]

$= c$ [$\because$ e is the identity element] …(iii)

Again $b * (a * c) = b * e$ [$\because$ $a * c = e$]

$= b$ [$\because$ e is the identity element] …(iv)

Since '*' is an associative binary operation on S.

Therefore, $(b * a) * c = b * (a * c) \Rightarrow$ $c = b$

Hence inverse of an element is unique if the binary operation is associative.

THEOREM 2. Let * be an associative binary operation on a set S and a be an invertible element of S. Then, $(a^{-1})^{-1} = a$.

***Proof*:**

Let e be the identity element in S for the binary operation * on S.

Now, $a * a^{-1} = e = e^{-1} * a$

$\Rightarrow$ $a^{-1} * a = e = a * a^{-1}$

$\Rightarrow$ a is inverse of a^{-1} $\Rightarrow$ $a = (a^{-1})^{-1}$.

11. BINARY COMPOSITION TABLES

Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set of n elements. A binary operation, say * on A can also be described by means of a table with n + 1 rows and n + 1 columns. The first row and first column contains the elements of the set A. The element $a_i * a_j$ of A is shown in the space at the intersection of (i + 1)th row and (j + 1) th column.

COMPOSITION TABLE

*	a_1	a_2	…	…	…	a_n
a_1	$a_1 * a_1$	$a_1 * a_2$				$a_1 * a_n$
a_2	$a_2 * a_1$	$a_2 * a_2$	…	…	…	$a_2 * a_n$
:						
:						

:						
a_n	$a_n * a_1$	$a_n * a_2$				$a_n * a_n$

For example,

COMPOSITION TABLE

*	*a*	*b*	*c*
a	*b*	*c*	*a*
b	*c*	*a*	*b*
c	*a*	*b*	*c*

For the set $A = \{a, b, c\}$.

$\therefore$ we have $a * a = b, \quad a * b = c, \quad a * c = a$

$b * a = c, \quad b * b = a, \quad b * c = b$

$c * a = a, \quad c * b = b, \quad c * c = c$

Illustration 11

Question: **Let *A* = {1, –1, *i*, –*i*}, where *i* Z √J1. Draw the composition table corresponding to binary operation 'multiplication' on *A*.**

Solution:

$1 \times 1 = 1,$	$1 \times (-1) = -1,$	$1 \times i = i,$	$1 \times (-i) = -i,$
$(-1) \times 1 = -1,$	$(-1) \times (-1) = 1,$	$(-1) \times i = -i,$	$(-1) \times (-i) = i,$
$i \times 1 = i,$	$i \times (-1) = -i,$	$i \times i = -1,$	$i \times (-i) = 1,$
$(-i) \times 1 = -i,$	$(-i)(-1) = i,$	$(-i) \times i = 1,$	$(-i) \times (-i) = -1.$

Hence the required composition table is :

×	1	–1	*i*	–*i*
1	1	–1	*i*	–*i*
–1	–1	1	–*i*	*i*
i	*i*	–*i*	–1	1
–*i*	–*i*	*i*	1	–1

Illustration 12

Question: *** is a binary operation defined on *Q*. Find which of the following binary operations are associative.**

(i) ***a* * *b*** Z ***a*** J ***b*** ? ***a*, *b*** Č ***Q***

(ii) ***a* * *b*** Z $\frac{ab}{2}$? ***a*, *b*** Č ***Q***

(iii) ***a * b*** Z **2*a*** H **3*b*** ? ***a*, *b*** Č ***R***

Solution: (i) $(a*b)*c=(a-b)*c=(a-b)-c=a-b-c$

$a*(b*c)=a*(b-c)=a-(b-c)=a-b+c\neq(a*b)*c$

Hence the operation * is not associative

(ii) $(a*b)*c=\left(\frac{ab}{2}\right)*c=\frac{abc}{4}$

$a*(b*c)=a*\left(\frac{bc}{2}\right)=\frac{abc}{4}=(a*b)*c$.

Hence the operation * is associative

(iii) $a*(b*c)=a*(2b+3c)=2a+3(2b+3c)=2a+6b+9c$

$(a*b)*c=(2a+3b)*c=2(2a+3b)+3c=4a+6b+3c$

$\therefore\quad a*(b*c)\neq(a*b)*c.$

Hence * is not associative.

Important formulae/points

- A binary operation (or binary composition) is denoted by * or $\odot$
- A binary operation * on *A* is a mapping *: $A \times A \to A$ defined by * (*a*, *b*) of *A*.
- If the set *A* has *m* elements, then number of binary operations on *A* is $(m)^{m^2}$.
- If *a* * *b* = *b* * *a* for *a*, *b* $\in$ *S*, then we say that the binary operation * is commutative.
- A binary operation '*' on a set *S* is said to be an associative binary operation, if

  $(a * b) * c = a * (b * c)$ for all *a*, *b*, *c* $\in$ *S*.

- '*' is said to be distributive over $\odot$, if

  $a * (b \odot c) = (a * b) \odot (a * c)$ and $(b \odot c) * a = (b * a) \odot (c * a)$ for all *a*, *b*, *c* $\in S$.
- An element $e \in A$ is called an identity element for the binary operation * if $e * a = a * e = a$ for all $a \in A$.
- Identity element for a binary operation if it exists is unique.
- An element $a \in A$ is said to be invertible with respect to binary operation * if there exists an element $b \in A$ such that $a * b = b * a = e$. The element is called on inverse of element *a* and is denoted by a^{-1}
- Let *Z* be the set of all integers and * be the operation of ordinary addition then $0 \in Z$ is the identity element.
- $(a^{-1})^{-1} = a$

PRACTICE PROBLEMS

PP1. Find the domain and range of the following relations:

(a) $R_1 = \{(1, 2), (1, 4), (1, 6), (1, 10)\}$ (b) $R_2 = \left\{\left(x, \frac{1}{x}\right) : 0 < x < 4, x \text{ is an integer}\right\}$

PP2. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the number of relations from A to B.

PP3. Let $A = \{a, b, c\}, B = \{x, y\}$. Find the total number of relations from A to B.

PP4. Test the following functions for one-one and onto:

(i) $f : N \to N : f(x) = 3x + 5$ (ii) $f : R \to R : f(x) = x^2$

(iii) $f : R \to R : f(x) = x^2 + 5$

where N = set of natural number, R = set of real numbers.

PP5. Let $f : R \to R : f(x) = (2x + 1)$ and let $g : R \to R : g(x) = (x^2 - 2)$.

Write down the formulae for:

(i) gof (ii) fog (iii) gog

PP6. If the mappings f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\};\ g = \{(2, 3), (5, 1), (1, 3)\}$, then define fog and gof.

PP7. Let $f : R \to R : f(x) = x^2; g : R \to R : g(x) = \tan x$ and $h : R \to R : h(x) = \log x$.

Find a formula for $ho(gof)$ and determine $[ho(gof)]\sqrt{\left(\frac{\pi}{4}\right)}$.

PP8. Let $f : R \to R$ be defined by $f(x) = 3x + 2$. Show that f is invertible. Find $f^{-1} : R \to R$

PP9. If $f : [0, \infty) \to [2, \infty)$ be defined by $f(x) = x^2 + 2, \forall\ x \in R$. Then find f^{-1}.

PP10. Let $A = R - \{3\}$ and $B = R - \{1\}$. If $f : A \to B : f(x) = \frac{x-2}{x-3}$, show that f is bijective.

PP11. Let $f : R \to R$ be defined by $f(x) = 2x - 3$ and $g : R \to R$ be defined by $g(x) = \frac{x+3}{2}$. Show that $fog = I_R = gof$.

PP12. Let $f : A \to B$ and $g : B \to C$ be two one-one functions. Show that gof is also one-one function.

PP13. * is a binary operation defined on Q. Find which of the binary operations are commutative

(i) $a * b = a^2 - b^2\ \forall\ a, b \in Q$ (ii) $a * b = 2a^2 + b^2\ \forall\ a, b \in Q$

(iii) $a * b = a + ab\ \forall\ a, b \in Q$ (iv) $a * b = (a + b)^2\ \forall\ a, b \in Q$

PP14. If * is a binary operation on the set R defined by $a * b = 3a + 4b\ \forall\ a, b \in R$, then show that * is not associative.

PP15. * is a binary operation defined on Q. Find which of the following binary operations are associative.

(i) $a * b = a + b\ \forall\ a, b \in Q$ (ii) $a * b = \frac{ab}{6}\ \forall\ a, b \in Q$

(iii) $a * b = a + b - ab\ \forall\ a, b \in Q$ (iv) $a * b = a^2 b\ \forall\ a, b \in Q$

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Show that the relation *R*, defined in the set *A* of all polygons as *R* = {(P_1, P_2) : (P_1 and P_2 have same number of sides), is an equivalence relation of *A*. What is the set of all elements in *A* related to the right angle triangle *T* with sides 3, 4 and 5?

Solution:

Given *A* = set of all polygons

$R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides}

(i) Reflexive: Let $P \in A$

Clearly, number of sides of *P* = number of sides of *P*

$\therefore\ (P, P) \in R, \forall P \in A$

Hence *R* is reflexive.

(ii) Symmetric: Let $P_1, P_2 \in A$

Let $(P_1, P_2) \in R \Rightarrow P_1$ and P_2 have same number of sides

$\Rightarrow P_2$ and P_1 have same number of sides

$\Rightarrow (P_2, P_1) \in A$

Hence *R* is symmetric.

(iii) Transitive: Let $P_1, P_2, P_3 \in A$

Let $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$

$\Rightarrow$ {number of sides of P_1 = number of sides of P_2

and number of sides of P_2 = number of sides of P_3}

$\Rightarrow$ Number of sides of P_1 = number of sides of P_3

$\Rightarrow (P_1, P_3) \in R$

Hence *R* is transitive.

Thus *R* is reflexive, symmetric and transitive and hence *R* is an equivalence relation on *A*.

***Example* 2:**

Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 and R_2 be relations in X given by $R_1 = \{(x, y) : x - y$ is divisible by 3$\}$ and $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$, where '$\subset$' means subset of. Show that $R_1 = R_2$.

Solution:

Given $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Let $A_1 \{1, 4, 7\}$, $A_2 = \{2, 5, 8\}$, $A_3 = \{3, 6, 9\}$

Clearly difference of any two elements of sets A_1 or A_2 or A_3 is a multiple of 3.

Now $(x, y) \in R_1 \Leftrightarrow x - y$ is a multiple of 3

$\Leftrightarrow (x, y)$ belong to the same set A_1 or A_2 or A_3

$\Leftrightarrow \{x, y\} \subset A_1$ or $\{x, y\} \subset A_2$ or $\{x, y\} \subset A_3$

$\Leftrightarrow (x, y) \in R_2$

Hence $R_1 = R_2$

Example 3:

Let *A* be a set having more than one element. Let '*' be a binary operation on *A* defined by $a * b = a$ for all $a, b \in A$. Is '*' **commutative or associative on *A*?**

Solution:

Let $a, b \in A$. Then

$a * b = a$ and $b * a = b$

$\therefore \quad a * b \neq b * a$

So, '*' is not commutative on *S*.

Again let $a, b, c \in A$. Then

$(a * b) * c = a * c = a$

and $a * (b * c) = a * b = a$

$\therefore \quad (a * b) * c = a * (b * c)$ for all $a, b, c \in A$

So, '*' is associative on *A*.

Thus '*' is associative on *A* but it is not commutative on *A*.

Example 4:

If $f(x) = e^x$ and $g(x) = \log_e x (x > 0)$, find *fog* and *gof*. Is $fog = gof$?

Solution:

We have, domain $(f) = R$, Range (f) = $(0, \infty)$; Domain $(g) = (0, \infty)$ and Range $(g) = R$

Computation of fog : We observe that

Range $(g)=(0,\ \infty)\subseteq$ Domain $(f)=R$

$\therefore$ fog exists and fog : Domain $(g)\to R$ i.e., $fog:(0,\ \infty)\to R$

Also, $fog(x)=f(g(x))=f(\log_e x)=e^{\log_e x}=x$

Thus $fog:(0,\ \infty)\to R$ is defined as $fog(x)=x$

Computation of gof : we have, Range $(f)=(0,\ \infty)$ = Domain (g)

$\therefore$ gof exists and gof : domain $(f)\to R$ i.e., $gof:R\to R$

Also $gof(x)=g(f(x))=g(e^x)=\log_e e^x=x\log_e e=x$

Thus $gof:R\to R$ is defined as $gof(x)=x$

We observe that domain $(gof)\neq$ domain (fog)

$\therefore$ $gof\neq fog$

***Example* 5:**

Show that $f : R-\{-1\} \to R-\{1\}$ given by $f(x)=\frac{x}{x+1}$ is invertible. Also find f^{-1}.

Solution:

In order to prove the invertibility of $f(x)$, it is sufficient to show that it is a bijection.

f is one-one: For any $x,\ y\in R-\{-1\}$, we have $f(x)=f(y)$

$\Rightarrow$ $\frac{x}{x+1}=\frac{y}{y+1}$ $\Rightarrow$ $xy+x=xy+y$

$\Rightarrow$ $x=y$

So, *f* is one-one.

f is onto: Let $y\in R-\{1\}$. Then $f(x)=y$

$\Rightarrow$ $\frac{x}{x+1}=y$ $\Rightarrow$ $x=\frac{y}{1-y}$

Clearly, $x\in R$ for all $y\in -\{1\}$

Also $x\neq -1$.

Because $x=-1$ $\Rightarrow$ $\frac{y}{1-y}=-1$ $\Rightarrow$ $y=-1+y$, which is not possible.

Thus, for each $y\in R-\{1\}$ there exists $x=\frac{y}{1-y}\in R-\{-1\}$ such that $f(x)=\frac{x}{x+1}=\frac{\frac{y}{1-y}}{\frac{y}{1-y}+1}=y$

So, *f* is onto.

Thus, *f* is both one-one and onto. Consequently it is invertible.

Now, $fof^{-1}(x) = x$ for all $x \in R - \{1\}$

$\Rightarrow \quad f\left(f^{-1}(x)\right) = x \;\Rightarrow\; \dfrac{f^{-1}(x)}{f^{-1}(x)+1} = x \;\Rightarrow\; f^{-1}(x) = \dfrac{x}{1-x}$ for all $x \in R - \{1\}$

***Example* 6:**

Consider $f : R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$, where R^+ is the set of all non-negative real numbers. Show that *f* is invertible with $f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$.

Solution:

Given $f : R \to [-5, \infty)$, by $f(x) = 9x^2 + 6x - 5$ …(i)

To test whether *f* is one-one:

Let $x_1, x_2 \in R_+$ such that $f(x_1) = f(x_2)$

Now $f(x_1) = f(x_2) \;\Rightarrow\; 9x_1^2 + 6x_1 - 5 = 9x_1^2 + 6x_2 - 5$

$\Rightarrow \quad 9\left(x_1^2 - x_2^2\right) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad x_1 - x_2 = 0 \qquad [\because x_1, x_2 \in R_+, \therefore 9(x_1 + x_2) + 6 \neq 0]$

$\Rightarrow \quad x_1 = x_2$

Hence *f* is one-one.

To test whether *f* is onto:

Let *y* be an arbitrary element of $[-5, \infty)$

Let $f(x) = y$

Now, $y = f(x) \;\Rightarrow\; y = 9x^2 + 6x - 5$

$\Rightarrow \quad 9x^2 + 6x - (5 + y) = 0 \;\Rightarrow\; x = \dfrac{-6 \pm \sqrt{36 + 36(5+y)}}{18}$

$\Rightarrow \quad x = \dfrac{-6 \pm 6\sqrt{6+y}}{18} \;\Rightarrow\; x = \dfrac{-1 \pm \sqrt{6+y}}{3}$

$\Rightarrow \quad x = \dfrac{-1 + \sqrt{6+y}}{3} = \dfrac{\sqrt{6+y} - 1}{3} \qquad [\because y \in [-5, \infty), \therefore y \geq 0]$

$\Rightarrow \quad x \in$ domain R_+

Hence *f* is onto.

To find f^{-1}:

Let $y = f(x) \Rightarrow x = \frac{\sqrt{6+y}-1}{3} \Rightarrow f^{-1}(y) = \frac{\sqrt{6+y}-1}{3}$

***Example* 7:**

If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, show that $f(x)+f(y) = f\left(\frac{x+y}{1+xy}\right)$.

Solution:

We have, $f(x)=\log\left(\frac{1+x}{1-x}\right)$

$$\therefore \quad f(x)+f(y)=\log\left(\frac{1+x}{1-x}\right)+\log\left(\frac{1+y}{1-y}\right)=\log\left[\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)\right]$$

$$=\log\left(\frac{1+x+y+xy}{1-x-y+xy}\right) \qquad \ldots(i)$$

and $$f\left(\frac{x+y}{1+xy}\right)=\log\left(\frac{1+\frac{x+y}{1+xy}}{1-\frac{x+y}{1+xy}}\right)=\log\left(\frac{1+xy+x+y}{1+xy-x-y}\right) \qquad \ldots(ii)$$

From (i) and (ii) $\Rightarrow$ $f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right)$

***Example* 8:**

Show that the function $f(x) = \frac{x^2-8x+18}{x^2+4x+30}$ is not one-one.

***Solution*:**

A function is one-one if $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ (only)

Now $f(x_1)=f(x_2)\Rightarrow \frac{x_1^2-8x_1+18}{x_1^2+4x_1+30}=\frac{x_2^2-8x_2+18}{x_2^2+4x_2+30}$

$\Rightarrow 12x_1^2x_2-12x_1x_2^2+12x_1^2-12x_2^2-312x_1+312x_2=0$

$\Rightarrow (x_1-x_2)\{12x_1x_2+12(x_1+x_2)-312\}=0$

$\Rightarrow x_1=x_2$ or $x_1=\frac{26-x_2}{1+x_2}$

Since $f(x_1)=f(x_2)$ does not imply $x_1=x_2$ alone, $f(x)$ is not a one-one function.

***Example* 9.**

If *fog* = |sin *x*| and *gof* = $\sin^2\sqrt{x}$, then find *f*(*x*) and *g*(*x*).

Solution:

fog = *f*(*g*(*x*)) = |sin*x*| = $\sqrt{\sin^2 x}$

Also *gof* = *g*(*f*(*x*)) = $\sin^2\sqrt{x}$

Obviously, $\sqrt{\sin^2 x} = \sqrt{g(x)}$ and $\sin^2\sqrt{x} = \sin^2(f(x))$

i.e. $g(x) = \sin^2 x$ and $f(x) = \sqrt{x}$.

***Example* 10:**

Find the range of the function $f(x) = \dfrac{x}{1+x^2}$.

Solution:

We have, $f(x) = \dfrac{x}{1+x^2}$

Since $f(x)$ is rational function and $1+x^2 \neq 0, \ \forall \ x \in R$

$\therefore \ y = f(x) \ \Rightarrow \ y = \dfrac{x}{1+x^2}$

$\Rightarrow \ x^2y - x + y = 0$

$\Rightarrow \ x = \dfrac{1 \pm \sqrt{1-4y^2}}{2y}$

Clearly, *x* will assume real values, if

$1-4y^2 \geq 0$ and $y \neq 0 \ \Rightarrow \ 4y^2 - 1 \leq 0$ and $y \neq 0$

$\Rightarrow \ y^2 - \dfrac{1}{4} \leq 0$ and $y \neq 0 \ \Rightarrow \ \left(y-\dfrac{1}{2}\right)\left(y+\dfrac{1}{2}\right) \leq 0$ and $y \neq 0$

$\Rightarrow \ y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right] - \{0\}$

Hence range $(f) = \left[-\dfrac{1}{2}, \dfrac{1}{2}\right] - \{0\}$

EXERCISE – I

1. If $f(x)=\frac{1}{2x+1}, x\neq -\frac{1}{2}$, then show that $f(f(x))=\frac{2x+1}{2x+3}$, provided that $x\neq -\frac{3}{2}$.

2. Let $f:[2,\infty)\to R$ and $g:[-2,\infty)\to R$ be two real functions defined by $f(x)=\sqrt{x-2}$ and $g(x)=\sqrt{x+2}$. Find $f+g$ and $f-g$.

3. If *f, g, h* are real functions defined by $f(x)=\sqrt{x}$, $g(x)=\frac{1}{x}$ and $h(x)=2x^2-3$, then find the values of $(2f+g-h)(1)$ and $(2f+g-h)(0)$.

4. Find the range of each of the following function.

(a) $f(x)=1-|x-2|$

(b) $f(x)=\sqrt{16-x^2}$

(c) $f(x)=\frac{3}{2-x^2}$

(d) $f(x)=\frac{1}{\sqrt{4+3\sin x}}$

5. Find the domain of each of the function given by

(a) $f(x)=\sin^2 x^3+\cos^2 x^3$

(b) $f(x)=\sin\left(\sqrt{4-x^2}\right)$

6. Let $f:[-1,\infty)\to[-1,\infty)$ is given by $f(x)=(x+1)^2-1, x\geq -1$. Show that f is invertible. Also find the set $S=\{x: f(x)=f^{-1}(x)\}$.

7. If $f(x)=\frac{x-1}{x+1}$, then prove that $f(2x)=\frac{3f(x)+1}{f(x)+3}$.

8. Which of the following functions are one-one?

(i) $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x)=-|x|$

(ii) $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x)=3x+2$

(iii) $f:\mathbf{R}^+\to\mathbf{R}^+$ defined by $f(x)=1+2x^2$

(iv) $f:\mathbf{R}-\{1\}\to\mathbf{R}$ defined by $f(x)=\frac{x+1}{x-1}$.

9. * is a binary operation defined on Q. Find which of the following binary operations is/are associative.

(i) $a*b=ab^2\ \forall\ a, b\in Q$

(ii) $a*b=a+b+ab\ \forall\ a, b\in Q$

10. Is the relation R defined on the set Q^* of non-zero rational number, by $xRy \Leftrightarrow xy = 1, \forall x, y \in Q^*$; an equivalence relation ?

11. Let N be the set of natural number. Let a relation R be defined on $N \times N$ by $(a, b) R (c, d) \Leftrightarrow ad = bc$, prove that R is an equivalence relation.

12. Let a relation R' in the set of real numbers be defined by $xR'y \Leftrightarrow 1 + xy > 0$. Show what R' is reflexive and symmetric but not transitive.

13. Let * be a binary operation on Z defined by $a * b = a + b - 4$ for all $a, b \in Z$.

(i) show that '*' is both commutative and associative

(ii) find the identity element in Z

(iii) find the invertible elements in Z

14. Let * be a binary operation defined on N such that $a * b = a^2 + b^2 + 2$ for $a, b \in N$:

(i) Find 2 * 5

(ii) show that $1 * 2 = 2 * 1$

(iii) show that '*' is commutative

15. If $a * b = 4a^2 + 6b^2$ be a binary operation on Q, then verify that * is not commutative.

EXERCISE – II

1. Find the domain of each of the following function given by

(a) $f(x)=\frac{1}{\sqrt{x+|x|}}$

(b) $f(x)=\frac{x}{x^2-3x+2}$

2. Let $f: N-\{1\}\to N$ defined by $f(n)$ = Highest prime factor of *n*, then show that *f* is neither one-one nor onto.

3. If $f(x)=\sqrt{x}(x\geq 0)$ and $g(x)=x^2-1$ are two real functions, find *fog* and *gof*. Is $fog=gof$?

4. Consider $f: R\to R$ given by $f(x)=4x+3$. Show that *f* is invertible. Find the inverse of *f*.

5. Let *N* be the set of all natural numbers. A relation *R* be defined on $N\times N$ by $(a, b)R(c, d)\Leftrightarrow a+d=b+c$. Show that *R* is an equivalence relation.

6. If *Q* is the set of rational numbers and *R* is a relation defined on *Q* by $xRy\Leftrightarrow |x-y|\leq\frac{1}{2}$, then prove that *R* is not an equivalence relation.

7. Show that the relation *R* in the set *A* of points in a plane given by *R* = (*P, Q*): distance of the point *P* from the origin is same as the distance of the point *Q* from the origin}, is an equivalence relation. Further, show that the set of all points related to a point $P\neq(0, 0)$ is the circle passing through *P* with origin as centre.

8. Show that the relation *R* defined in the set *A* of all triangles as $R=\{T_1, T_2\}: T_1$ is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

9. Prove that:

(i) the inverse of an equivalence relation is also an equivalence relation.

(ii) the intersection of two equivalence relations is also an equivalence relation.

(iii) the union of two symmetric relations is also symmetric.

10. Let '*' be a binary operation on *N*, the set of natural numbers defined by $a*b=a^b$ for all $a,b\in N$.

Is '*' associative or commutative on *N*?

11. Let '*' be a binary operation on N given by $a * b = L.C.M.(a, b)$ for all $a, b \in N$.

(i) Find the identity element in N

(ii) Which elements of N are invertible? Find them.

12. Consider the set S = {1, 2, 3, 4}. Define a binary operation * on S as follows $a * b = r$, where r is the least non-negative remainder when ab is divided by 5. Construct the composition (operation) table for '*' on S.

13. If $f : N \to N, g : N \to N$ and $h : N \to R$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$, $\forall$ x, y and z in N. Show that $ho(gof) = (hog)of$.

14. If $f : A \to B$ and $g : B \to A$ satisfy $fog = I_B$. Show that f is onto and g is one-one.

15. Consider $f : R_+ \to [-3, \infty)$ given by $f(x) = 9x^2 + 6x - 3$. Show that f is inventible. Find inverse of f.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. (a) Domain = {1}, Range = {2, 4, 6, 10}

(b) Domain R = {1, 2, 3}, Range R = $\left\{1, \frac{1}{2}, \frac{1}{3}\right\}$

PP2. 16

PP3. 64

PP4. (i) one-one into (ii) neither one-one nor onto (iii) many-one into

PP5. (i) $(2x+1)^2-2$ (ii) $2(x^2-2)+1$ (iii) $(x^2-2)^2-2$

PP6. $fog = \{(2, 5), (5, 2), (1, 5)\}$

$gof = \{(1, 3), (3, 1), (4, 3)\}$

PP7. (i) $ho(gof) = \log\tan x^2$

(ii) 0

PP8. $f^{-1}(x) = \frac{x-2}{3}$

PP9. $f^{-1}(x) = \sqrt{x-2}$

PP13. (iv)

PP15. (i), (ii), (iii)

EXERCISE – I

2. $(f+g)x = \sqrt{x-2} + \sqrt{x+2} \quad \forall \quad x \in [2, \infty)$

$(f-g)x = \sqrt{x-2} - \sqrt{x+2} \quad \forall \quad x \in [2, \infty)$

3. 4, does not exist

4. (a) $(-\infty, 1]$ (b) $[0, 4]$

(c) $(-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$ (d) $\left[\frac{1}{\sqrt{7}}, 1\right]$

5. (a) Domain is R
(b) Domain is $[-2, 2]$

6. $S = \{-1, 0\}$

8. (ii), (iii) and (iv) are one-one.

9. (ii)

10. No, as it is not reflexive

11. No

13. (ii) 4 (iii) All elements in Z are invertible; inverse of a is $8 - a$

14. (i) 31

EXERCISE – II

1. (a) $(0, \infty)$

(b) $R - \{1, 2\}$

3. (i) $fog(x) = \sqrt{x^2 - 1}, gof(x) = x - 1, fog \neq gof$.

4. $f^{-1}(x) = \frac{x-3}{4}$

10. '*' is neither commutative nor associative on N

11. (i) 1 (ii) 1

12.

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

15. $f^{-1}(x) = \left(\frac{\sqrt{x+4} - 1}{3}\right)$