LESSON 15

# PROBABILITY

## 1. INTRODUCTION

Some events seem to have a certainty about their outcome; while a few are certain not to happen. There are others, which, with regard to their outcome, vary between the two extreme situations referred to above. In a rough way, a measure of the extent of the happening or non-happening of an event may be said to be given by the term **Probability.** The word probability and the word chance are synonymous and may be taken, in this context, to be indistinguishable.

Probability, in the conventional sense, is a ratio between what may be called as the 'number of favourable' to the 'total number'; and, as such this ratio is a positive fraction varying in value between zero (when the events is certain not to happen) to one (when an event is certain to happen).

Thus the probability of having a white ball from a bag containing all black balls is zero, while that of drawing a black is unity.

## 2. CONCEPT OF PROBABILITY IN SET THEORETIC LANGUAGE

### 2.1 RANDOM EXPERIMENT

It is an operation which can result in any one of its well defined outcomes and the outcome cannot be predicted with certainity.

For example some of the random experiments are

- toss of a coin, which can result in either a head or a tail
- throw of a die which can result in any one of the six faces
- drawing a card from a pack of 52 cards which can result in any one of the 52 cards.

## 2.2 SAMPLE SPACE AND SAMPLE POINTS

The set of all possible outcomes of a random experiment is called sample space and is denoted by $S$. Every possible outcome i.e. every element of this set is a sample point.

For example:

- In toss of a coin, $S = \{H, T\}$ where $H$ and $T$ are sample points representing a head and a tail respectively.
- In throw of a die, $S = \{1, 2, 3, 4, 5, 6\}$ where the numbers are the sample points representing the six faces.
- **Discrete Sample Space:** A sample space having finite number of sample points, is said to be 'Discrete Sample Space'. All the above examples of sample space are discrete sample space. While a sample space containing non-enumerable number of points is said to be a continuous space.

***Illustration* 1**

***Question:*** **A dice is thrown twice. Describe the sample space of this experiment.**

***Solution:*** When a dice is tossed once, there are six possible outcomes as the uppermost face of the dice may show up any one of the numbers from 1 to 6.

This means that when the dice is thrown twice, the sample space contains the outcomes of the type:

'*a* number *x* on first throw and a number *y* on second throw', where *x, y* are natural numbers from the set {1, 2, 3, 4, 5, 6}.

So the random experiment 'throwing a dice twice' results into the sample space

$S = \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\}$

i.e., $S = \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$.

***Illustration* 2**

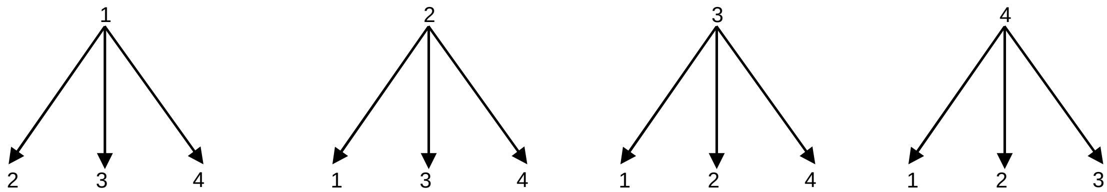
***Question:*** **The number 1, 2, 3 and 4 are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.**

***Solution:*** Here, the first slip has four possibilities and second only three.

So, there are $4 \times 3 = 12$ possible outcomes.

The second slip must show a number different from one on the first slip.

First slip




Second slip

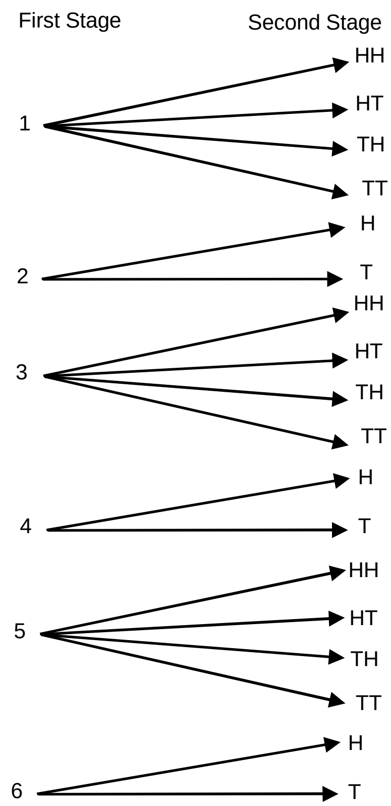
If $S$ is the sample space, then

$$S = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}.$$

***Illustration* 3**

***Question:*** **An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.**

***Solution:*** It is a two stage experiment. When a die is rolled, it may show up any one of the six numbers 1, 2, 3, 4, 5, 6.




When the dice shows up an odd number, a coin is tossed twice and hence, corresponding to each of 1, 3, and 5 there are four possibilities.

When the dice shows up an even number, a coin is tossed once and hence, corresponding each of 2, 4, and 6, there are only two possible outcomes. In this case, the sample

Space is $S = \left\{ \begin{array}{llllll} 2H, & 2T, & 4H, & 4T, & 6H, & 6T, \\ 1HH, & 1HT, & 1TH, & 1TT, & 3HH, & 3HT, \\ 3TH, & 3TT, & 5HH, & 5HT, & 5TH, & 5TT \end{array} \right\}$

Note that there are 18 possible outcomes.

## 2.3 TRIAL

When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a trial and the outcomes are called cases. The number of times the experiment is repeated is called the number of trials.

For example:

- One toss of coin is a trial when coin is tossed 5 times.
- One throw of a die is a trial when the die is thrown 4 times.

## 2.4 EVENT

A subset of sample space, i.e. a set of some of possible outcomes of a random experiment is called as event.

Consider an experiment in which a pair of dice are thrown. The sample space *S* of this experiment consists of 36 points.

*S* = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

*A* : Event having "sum of numbers on the faces is 9"

= {(6, 3), (5, 4), (4, 5), (3, 6)}

*B* : Event having "score of the number appeared is > 12" equals $\phi$ (null event)

**Simple event**

Each sample point in the sample space is called an elementary event or simple event. For example occurrence of head in throw of a coin is simple event.

**Sure event**

The set containing all sample points is a sure event as in the of a throw die the occurrence of natural number less than 7, is a sure event.

**Null event**

The set which does not contain any sample point.

**Mixed/compound event**

A subset of sample space *S* containing more than one element is called a mixed event or a compound event.

In case of throwing a die, appearing of odd numbers up is a Mixed Event.

*E* = {1, 3, 5} which has three elements.

**Complement of an event**

Let *S* be the sample space and *E* be an event then $E^c$ or $\overline{E}$ or $E'$ represents complement of event *E* which is a subset containing all sample points in *S* which are not in *E*. It refers to the non occurrence of event *E*.

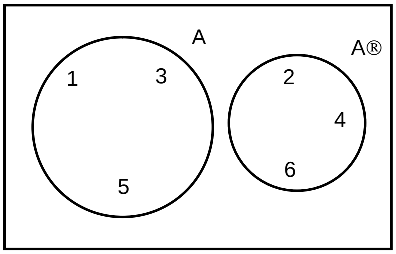
For a single throw of a die

*S* = {1, 2, 3, 4, 5, 6}

if *A* = {1, 3, 5}

then $A'$ = {2, 4, 6}

By pictorial representation,




$A \cap A' = \emptyset$

$P(A \cap A') = 0$

## 2.5 ALGEBRA OF EVENTS

In connection with basic probability laws we shall need the following concepts and facts about events (subsets) *A*, *B*, *C*, …… of a given sample space *S*.

**The union $A \cup B$ of A and *B* consists of all points in A or *B* or both.**

**The intersection $A \cap B$ of A and *B* consists of all points that are in both *A* and *B*.**

If *A* and *B* have no points in common, we write

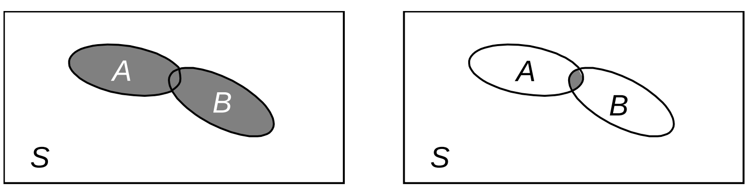
$$A \cap B = \emptyset$$

where $\emptyset$ is the empty set (set with no elements) and we call *A* and *B* mutually exclusive (or disjoint) because the occurrence of *A* excludes that of *B* (and conversely) for example, if your die turns up an odd number, it cannot turn up an even number in the same trial. Similarly, a coin cannot turn up Head and Tail at the same time.

The complement $A^c$ of A consists of all the points of *S* not in *A*. Thus,

$$A \cap A^c = \emptyset, \quad A \cup A^c = S$$

Working with events can be illustrated and facilitated by Venn diagrams for showing union, intersections, and complements, as shown in the figure.




Union $A \cup B$ Intersection $A \cap B$

*Venn diagrams showing two events A and B in a sample space S*

*and their union $A \cup B$ (coloured) and intersection $A \cap B$ (coloured)*

Union and intersections of more events are defined similarly. The union

$$\bigcup_{j=1}^{m} A_j = A_1 \cup A_2 \cup \ldots\ldots \cup A_m$$

of events $A_1$ , ……… , $A_m$ consists of all points that are in at least one $A_j$. Similarly for the union $A_1 \cup A_2 \cup$ ….. of infinitely many subsets $A_1$ , $A_2$ , ……. of an infinite sample space *S*(that is, *S* consists of infinitely many points). The intersection

$$\bigcap_{j=1}^{m} A_j = A_1 \cap A_2 \cap \ldots\ldots \cap A_m$$

of $A_1$ , ……. , $A_m$ consists of the points of $S$ that are in each of these events. Similarly for the intersection $A_1 \cap A_2 \cap$ ……. of infinitely many subsets of $S$.

| Verbal description of the event | Equivalent set theoretic notation |
|---|---|
| Not *A* | $\overline{A}$ |
| *A* or *B* (at least one of *A* or *B*) | $A \cup B$ |
| *A* and *B* | $A \cap B$ |
| *A* but not *B* | $A \cap \overline{B}$ |
| Neither *A* nor *B* | $\overline{A} \cap \overline{B}$ |
| At least one of *A, B* or *C* | $A \cup B \cup C$ |
| Exactly one of *A* and *B* | $(A \cap \overline{B}) \cup (\overline{A} \cap B)$ |
| All three of *A, B* and *C* | $A \cap B \cap C$ |
| Exactly two of *A, B* and *C* | $(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$ |

## 2.6 EQUALLY LIKELY EVENTS

The events are said to be equally likely if none of them is expected to occur in preference to the other one. For example

In throw of a fair coin occurrence of a head or a tail have equal chances. Hence event that a head appears and event that a tail appears are equally likely events.

## 2.7 MUTUALLY EXCLUSIVE EVENTS

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any other. For example:

- In throw of a die, the event of occurrence of an even number and the event of occurrence of an odd number are mutually exclusive.
- In throw of a fair coin, occurrence of a head or a tail are mutually exclusive.
- Consider the experiment of throwing a die. Let *A* be the event "the number appeared is greater than zero but less than 4". Then $A = \{1, 2, 3\}$. Let *B* be the event, the number appeared is at least 5. Then $B = \{5, 6\}$.

  Clearly, $A \cap B = \emptyset$

The joint occurrence of *A* and *B* is thus an impossible event. The event *A* and *B* are said to be "Mutually Exclusive Events".

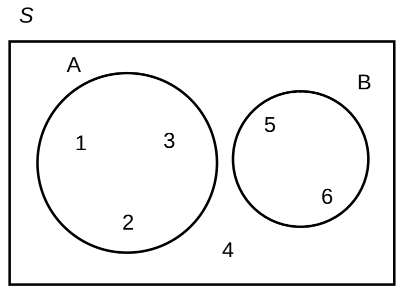



***Illustration 4***

***Question:*** **Three coins are tossed once. Let *A* denote the event 'three heads show'; *B* denote the event 'two heads and one tail show'; *C* denote the event 'three tails show' and *D* denote the event 'a head shown on the first coin'. Which events are**

**(i) mutually exclusive? (ii) simple? (iii) compound?**

***Solution:*** Here, the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

When three coins are tossed once or one coin is tossed thrice, the sample space is same.

Here, *A* : 'three heads show up'

i.e., *A* = {HHH}, *B* : 'two heads and one tail show'

i.e., *B* = {*HTH, HHT, THH*}, *C* : 'three tails show'

i.e., *C* = {*TTT*} and *D* : 'a head shown on the first coin,

i.e., *D* = {*HHH, HHT, HTH, HTT*}

(i) Here, $A \cap B = A \cap C = B \cap C = D \cap C = \phi$

Hence, *A* and *B* are mutually exclusive,

*A* and *C* are mutually exclusive,

*B* and *C* are mutually exclusive

and *C* and *D* are mutually exclusive.

(ii) As *A* and *C* contains only one outcome, therefore, *A, C* are simple event.

(ii) Both *B* and *D* are compound events as either of them contains more than one outcomes.

## 2.8 EXHAUSTIVE EVENTS

A set of events is exhaustive if the performance of the experiment results in occurrence of at least one of them. For example:

- In throw of a die, the event of occurrence of an even numbers and the event of occurrence of an odd number are exhaustive.
- In case of one, two and three tosses of a fair coin, the exhaustive set of events are respectively.

$$\{H, T\} \quad \begin{matrix} HH & HT \\ TH & TT \end{matrix} \quad \text{and} \quad \begin{matrix} HHH & HHT \\ HTH & THH \\ HTT & THT \\ TTH & TTT \end{matrix}$$

(one coin) (two coins) (three coins)

Exhaustive events cover the whole of the sample space. Their union is equal to $S$.

**Illustration 5**

***Question:*** **Two dice are rolled. '*A*' is the event that the sum of the numbers shown on the two dice is 5. '*B*' is the event that atleast one of the dice shows up 3. Are the two events *A* and *B*:**

**(i) mutually exclusive** **(ii) exhaustive?**

**Give arguments in support of your answer.**

***Solution:*** Here, the sample space is $S = \{(x, y) : x, y \in \{1, 2, 3, 4, 5, 6\}\}$,

which contains 6 × 6 = 36 equally likely simple events.

Now, '$A$' : "the sum of the numbers shown on the two dice is 5"

and '$B$' : "atleast one of the dice show up 3",

i.e., $A$ = {1, 4), (4, 1), (2, 3), (3, 2)}

and $B$ = {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (1, 3), (2, 3), (4, 3), (5, 3), (6, 3)

Since $A \cap B = \{(2, 3), (3, 2)\} \neq \phi$, therefore, $A$ and $B$ are not mutually exclusive.

Also, $A \cup B \neq S$, therefore, $A$ and $B$ are not exhaustive.

**_Illustration_ 6**

***Question:*** **Let $E_1$, $E_2$ and $E_3$ be three events associated with the sample space $S$ of a random experiment. If $E_1$, $E_2$ and $E_3$ also denote the subsets of $S$ representing these events, what are the sets representing the events:**

**(a) Out of the three events at least two events occur.**

**(b) Out of the three events only one occurs.**

**(c) Out of the three events only $E_1$ occurs.**

**(d) Out of three events not more than two occur.**

**(e) Out of three events exactly two events occur.**

**(f) None of the three events occur.**

***Solution:***

(a) 'Out of the three events, at least two events occur' means that either all the three events occur or two of them occur and the third does not. So, the set describing this event is $(E_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap E_3^c) \cup (E_1 \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3)$

(b) 'Out of the three events only one occurs' means that $E_1$ occurs and $E_2$, $E_3$ does not; $E_2$ occurs and $E_3$, $E_1$ does not; $E_3$ occurs and $E_2$, $E_1$ does not. So the set describing this event is

$$(E_1 \cap E_2^c \cap E_3^c) \cup (E_1^c \cap E_2 \cap E_3^c) \cup (E_1^c \cap E_2^c \cap E_3)$$

(c) 'Out of the three events only $E_1$ occurs' means that $E_1$ occurs and $E_2$ and $E_3$ both do not occur. So, the set describing this event is $E_1 \cap E_2^c \cap E_3^c$.

(d) 'Out of the three events not more than two occur' means that all the three events should not occur simultaneously. So, the set describing this event is $(E_1 \cap E_2 \cap E_3)^c$.

(e) 'Out of the three events exactly two events occur' means that two of the three events occur and the third does not. So, the set describing this event is

$$(E_1 \cap E_2 \cap E_3^c) \cup (E_1 \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3)$$

(f) 'None of the three events occurs' means that neither $E_1$, nor $E_2$ and nor $E_3$ occurs. So, the set describing this event is $E_1^c \cap E_2^c \cap E_3^c$.

## 3. DEFINITION OF PROBABILITY WITH DISCRETE SAMPLE SPACE

If the sample space $S$ of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability of occurrence of an event $A$ is

$$P(A) = \frac{\text{Number of sample points in A}}{\text{Number of sample points in S}}$$

$$P(A) = \frac{n(A)}{n(S)}$$

In particular $P(S) = 1$ and $0 \leq P(A) \leq 1$.

**Illustration 7**

***Question:*** **Ten items out of a set of 100 are defective. What is the probability that 3 out of any four chosen are defective?**

***Solution:*** Probability $= \frac{{}^{90}C_1 \cdot {}^{10}C_3}{{}^{100}C_4} = \frac{144}{52283}$

**Illustration 8**

***Question:*** **Seven persons are to be seated on one side of a straight table. What is the probability that two particular persons will be seated next to each other ?**

***Solution:*** Total number of ways of 7 persons being seated is ${}^7P_7 = |\underline{7}$ ways

If two are to be seated next to each other, treat them as one unit and this one unit with the remaining 5 can be seated in $|\underline{6}$ ways and in each one of these $|\underline{6}$ ways the two persons can be interchanged in 2 ways.

$\therefore$ Probability $= \frac{2|\underline{6}}{|\underline{7}} = \frac{2}{7}$

## 4. AXIOMATIC DEFINITION OF PROBABILITY

Given a sample space $S$, with each event A of $S$, there is associated a number $P$(A), called the probability of A, such that the following axioms of probability are satisfied.

- For every A in $S$, $0 \leq P(A) \leq 1$
- The entire sample space has the probability $P(S) = 1$
- For mutually exclusive events $A$ and $B$ ($A \cap B = \phi$), $P(A \cup B) = P(A) + P(B)$.

## 5. BASIC THEORIES OF PROBABILITY

**5.1** For an event A and its complement $A^c$ in sample space $S_1$ $\boldsymbol{P(A^c) = 1 - P(A)}$

$\because$ $A \cap A^c = \phi$ and $A \cup A^c = S$

and $P(A \cup A^c) = P(A) + P(A^c)$

$P(S) = P(A) + P(A^c)$

$1 = P(A) + P(A^c)$

***Illustration* 9**

***Question:*** **A cricket club has 15 members, among whom only 5 can bowl. What is the probability of forming a team of 11 to consist at least 3 bowlers?**

***Solution:*** Total number of ways of forming the team $= {}^{15}C_{11} = {}^{15}C_4$

Of these, number of ways of formation of the team

(i) with one bowler $= {}^{5}C_1 \cdot {}^{10}C_{10} = 5$

(ii) with two bowlers $= {}^{5}C_2 \cdot {}^{10}C_9 = 100$

Probability that at least 3 bowlers are in the team $= 1 - \frac{105}{{}^{15}C_4} = \frac{12}{13}$

***Illustration* 10**

***Question:*** **Five coins are tossed simultaneously. Find the probability of event that atleast one head turns up. (Assume that coins are fair)**

***Solution:*** Let A be the event that 'at least one head turns up' since each coin turns up on either a head or a tail hence, the sample space consists of $2^5 = 32$ outcomes. Each outcome having a probability of occurrence as $\frac{1}{32}$. Then $A^c$ is the event that 'No head turns up'. Thus $A^c$ consists of only one outcome,

Hence $P(A^c) = \frac{1}{32}$

$$P(A) = 1 - \frac{1}{32} = \frac{31}{32}$$

## 5.2 ADDITION THEOREM OF PROBABILITY

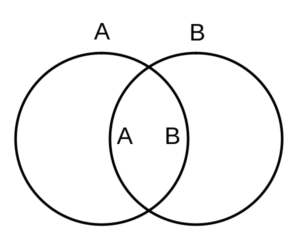
**If *A* and *B* be any two events in a sample space *S*, then the probability of occurrence of at least one of the events *A* and *B* is given by**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

***Proof:***

From set theory, we know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Dividing both sides by $n(S)$, we get




$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

or $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- If $A$ and $B$ are mutually exclusive, events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0$

$\therefore \quad P(A \cup B) = P(A) + P(B)$

- Two event $A$ and $B$ are mutually exclusive if and only if

$$P(A \cup B) = P(A) + P(B)$$

- $1 = P(S) = P(A \cup A') = P(A) + P(A')$ $\quad$ [$\because A \cap A' = \phi$]

or $P(A') = 1 - P(A)$

**Theorem: If *A*, *B* and *C* are any three events in a sample space *S*, then**

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

***Proof:***

From set theory, we know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Now, $n(A \cup B \cup C) = n[A \cup (B \cup C)]$

$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$

$= n(A) + n(B \cup C) - n[(A \cap B) \cup (A \cap C)]$

[$\because A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$]

$= n(A) + n(B \cup C) - n(D \cup E)$, where $D = A \cap B$ and $E = A \cap C$

$= n(A) + n(B \cup C) - [n(D) + n(E) - n(D \cap E)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(D) - n(E) + n(D \cap E)$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

[$\because D \cap E = (A \cap B) \cap (A \cap C) = A \cap B \cap C$]

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Dividing both sides by $n(S)$, we get

$$\frac{n(A \cup B \cup C)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} + \frac{n(C)}{n(S)} - \frac{n(A \cap B)}{n(S)} - \frac{n(B \cap C)}{n(S)} - \frac{n(A \cap C)}{n(S)} + \frac{n(A \cap B \cap C)}{n(S)}$$

$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

- If *A, B, C* are mutually exclusive events, then

$A \cap B = \phi,\ B \cap C = \phi,\ A \cap C = \phi,\ A \cap B \cap C = \phi$

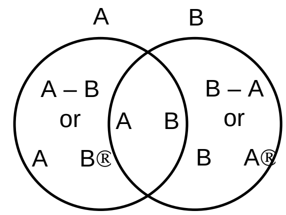
$\therefore\ P(A \cup B \cup C) = P(A) + P(B) + P(C)$

- If $A$ and $B$ are any two events, then

$(A - B) \cap (A \cap B) = \phi$ and $A = (A - B) \cup (A \cap B)$

$\therefore\ P(A) = P(A - B) + P(A \cap B)$

$= P(A \cap B') + P(A \cap B)$




or $P(A) - P(A \cap B) = P(A - B) = P(A \cap B')$ $\quad [\because A - B = A \cap B']$

Similarly, $P(B) - P(A \cap B) = P(B - A) = P(B \cap A')$

**Illustration 11**

**Question:** **The probability that at least one of $A$ and $B$ occurs is 0.6. If $A$ and $B$ occur simultaneously with probability 0.2. Find $P(\bar{A}) + P(\bar{B})$.**

**Solution:** Given $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$

Now, $P(\bar{A}) + P(\bar{B}) = 1 - P(A) + 1 - P(B)$

$= 2 - [P(A) + P(B)]$

$= 2 - [P(A) + P(B) - P(A \cap B)] - P(A \cap B)$

$= 2 - P(A \cup B) - P(A \cap B)$

$= 2 - 0.6 - 0.2 = 1.2$

**Illustration 12**

**Question:** **A card is drawn at random from a well shuffled pack of 52 cards. Find the probability that it is either a king or a spade.**

**Solution:** In this case, the sample contains 52 equally likely simple events each with probability $\frac{1}{52}$.

Let $E_1$ : 'a king is drawn' and $E_2$ : 'a spade is drawn, then

$P(E_1) = \frac{4}{52} = \frac{1}{13}$ and $P(E_2) = \frac{13}{52} = \frac{1}{4}$

($\because$ there are four kings and thirteen cards in the suit of spades)

Observe that $E_1$ and $E_2$ are not M.E.

Here $E_1 \cap E_2$ : 'king of spades is drawn' and hence, $P(E_1 \cap E_2) = \dfrac{1}{52}$

Required probability = $P$ (either a king or a spade is drawn)

$$= P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4+13-1}{52} = \frac{4}{13}$$

***Illustration* 13**

***Question:*** **In tossing a fair die, what is the probability of getting an odd number or a number less than 4?**

***Solution:*** $S = \{1, 2, 3, 4, 5, 6\}$

Let $A$ be the event that odd number occurs then $P(A) = \dfrac{3}{6} = \dfrac{1}{2}$ as A = {1, 3, 5}.

Let $B$ be the event that a number less than 4 occurs then $B = \{1, 2, 3\}$ and $P(B) = \dfrac{3}{6} = \dfrac{1}{2}$

then $A \cap B = \{1, 3\}$ (odd number less than 4)

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$$

**5.3** For exhaustive events $A_1, A_2, \ldots A_n$ in a sample space.

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1$$

**5.4** For mutually exclusive events $A_1 A_2 \ldots A_n$ in a sample space $S$

$$P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

**5.5** For mutually exclusive and exhaustive events $A_1, A_2, \ldots, A_n$ in a sample space.

$$P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = 1$$

**5.6** For an event $A$ in sample space $S$, **'the odds in favour of *A*'** are $\dfrac{P(A)}{P(A^c)}$ where $A^c$ is the complement of the event $A$. Also **'the odds against *A*'** are $\dfrac{P(A^c)}{P(A)}$.

For example, in throw of a die the odds in favour of "a multiple of 3 occurs" is 2 : 4 i.e. 1 : 2.

***Illustration* 14**

***Question:*** **If the probability that on any workday a garage will get 10–20, 21–30, 31–40, over 40 cars to service is 0.20, 0.35, 0.25, 0.12 respectively. What is the probability that on a given workday the garage gets atleast 21 cars to service?**

***Solution:*** Since these are mutually exclusive events. Hence required probability is

$0.35 + 0.25 + 0.12 = 0.72$.

**Illustration 15**

***Question:*** **There are four events $E_1$, $E_2$, $E_3$ and $E_4$ one of which must and only one can happen. The odds are 2 : 5 in favour of $E_1$, 3 : 4 in favour of $E_2$ and 3 : 1 against $E_3$. Find the odds against $E_4$.**

***Solution:*** We are given that the four events $E_1$, $E_2$, $E_3$ and $E_4$ are mutually exclusive and exhaustive.

Hence, $P(E_1) + P(E_2) + P(E_3) + P(E_4) = 1$ …(i)

Since odds in favour of $E_1$ are 2 : 5, therefore, $P(E_1) = \frac{2}{2+5} = \frac{2}{7}$ …(ii)

Since odds in favour of $E_2$ are 3 : 4, therefore , $P(E_2) = \frac{3}{3+4} = \frac{3}{7}$ …(iii)

Again odds against $E_3$ are 3 : 1, therefore, odds in favour of $E_3$ are 1 : 3

$P(E_3) = \frac{1}{1+3} = \frac{1}{4}$ …(iv)

From (i), (ii), (iii) and (iv), we have

$\frac{2}{7} + \frac{3}{7} + \frac{1}{4} + P(E_4) = 1 \qquad P(E_4) = 1 - \frac{2}{7} - \frac{3}{7} - \frac{1}{4} = \frac{28-8-12-7}{28} = \frac{1}{28}$

odds against $E_4$ are $1 - \frac{1}{28} : \frac{1}{128}$, i.e. 27 : 1

($\because P(E) = p$ odds against $E$ are $(1-p) : p$)

**Illustration 16**

***Question:*** **In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.**

***Solution:*** *P*(student is studying Mathematics or Biology)

= *P*(student is studying Mathematics) + *P*(student is studying Biology)

–*P*(student is studying both Mathematics and Biology)

($\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$)

$= \frac{40}{100} + \frac{30}{100} - \frac{10}{100} = \frac{60}{100} = 0.6$

## PRACTICE PROBLEMS

**PP1.** 2 boys and 2 girls are in room *X* and 1 boy and 3 girls in room *Y*. Specify the sample space for the experiment in which a room is selected and then a person.

**PP2.** A die is thrown repeatedly until a six comes up. What is the sample space for this experiment?

**PP3.** A die is rolled. Let *E* be the event "die shows 4" and *F* be the event "die shows even number". Are *E* and *F* mutually exclusive?

**PP4.** A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed, find the probability that the sum of numbers that turn up is (i) 3 (ii) 12.

**PP5.** In a lottery, a person chooses six different natural numbers at random from 1 to 20 and if these six numbers match with the six numbers already fixed by the lottery committee he wins the prize. What is the probability of winning the prize in the game.[order of the numbers is not important]

**PP6.** If a leap year is selected at random, what is the chance that it will contain 53 Tuesdays?

**PP7.** 6 boys and 6 girls sit in a row randomly, find the probability that all the 6 girls sit together.

**PP8.** There are 4 envelopes corresponding to 4 letters. If the letters are placed in the envelopes at random, what is the probability that all the letters are not placed in the right envelopes?

**PP9.** Two dice are tossed once. Find the probability of getting an even number on first die, or a total of 8.

**PP10.** The odds in favour of standing first of three students appearing at an examination are 1 : 2, 2 : 5 and 1 : 7 respectively. What is the probability that either of them will stand first.

# SOLVED SUBJECTIVE EXAMPLES

### *Example* 1:

**A bag '*A*' contains 3 White and 2 Black balls. A bag '*B*' contains 2 white and 4 Black balls. First a bag is chosen and then a ball is drawn. What is the probability that it is White?**

### *Solution:*

Probability of taking the first bag and then a white ball from it = $\frac{1}{2}\cdot\frac{3}{5}=\frac{3}{10}$

Probability of taking the $2^{nd}$ bag and drawing a white ball from it = $\frac{1}{2}\cdot\frac{2}{6}=\frac{1}{6}$

The two events are exclusive.

Probability of one of them happening $P\ (A\cup B)=P\ (A)+P\ (B)=\frac{3}{10}+\frac{1}{6}=\frac{7}{15}$

**Note:** Since the events are mutually exclusive, $A\cap B=\phi$ and $P\ (A\cap B)=0$

### *Example* 2:

**Each packet of certain items contains a coupon, which is equally likely to bear the letters A, N, S, H or U. If m packets are purchases, find the probability that the coupons can not be used to spell ANSHU.**

### *Solution:*

Let $E_1$ be the event that A is not present,

$E_2$ be the event that N is not present,

$E_3$ be the event that S is not present,

$E_4$ be the event that H is not present,

$E_5$ be the event that U is not present,

Then required probability

$$=\frac{\text{Number of favourable cases}}{\text{Total number of cases}}$$

$$=\frac{{}^5C_1.4^m-{}^5C_2.3^m+{}^5C_3.2^m-{}^5C_4.1^m+0}{5^m}$$

### *Example* 3:

**A black die and a red die are rolled. Find the probability that**

**(i) the sum of their scores is divisible by 5.**

**(ii) the sum of their scores is 8; given that at least one die shows a 3 or 4.**

### *Solution:*

In any one case the total number of ways of combining the six faces of one with the six faces of the other is 36. Now with regard to favourable cases.

(i) The sum of their scores is divisible by 5. Such ordered pairs are

(1, 4); (2, 3); (3, 2); (4, 1); (5, 5); (4, 6); (6, 4)

Probability is $\frac{7}{36}$

(ii) The sum of their scores is 8 given that at least one has a 3 or 4. The possible cases are

| **Red** | **Black** | **No. of cases** |
|---|---|---|
| 3 | 1, 2, 3, 4, 5, or 6 | 6 |
| 4 | 1, 2, 3, 4, 5, or 6 | 6 |
| 1, 2, 5, or 6 | 3 | 4 |
| 1, 2, 5, or 6 | 4 | 4 |

Thus there are totally 20 cases of which a total 8 is had is (3, 5), (4, 4), (5, 3)

$\therefore$ Required probability $= \frac{3}{20}$

### *Example* 4:

**A set of 3 numbers are chosen from the set of numbers 1, 2, 3, …, (2*n* + 1). What is the probability that the numbers chosen are in A.P.?**

### *Solution:*

Any three can be chosen in ${}^{(2n+1)}C_3$ ways. Regarding the favourable cases; if three numbers *a*, *b*, *c* chosen are to be in A.P. then $a + c = 2b$. Thus the sum of the extremes being 2*b* should be always even. The two extreme numbers chosen are both odd there are (*n* + 1) odd numbers or both even there are *n* even numbers.

Required probability is $\left({}^{(n+1)}C_2 + {}^{n}C_2\right) / \ {}^{(2n+1)}C_3$

$$= \frac{\frac{(n+1)\,n}{2} + \frac{n(n-1)}{2}}{\frac{(2n+1)\,2n(2n-1)}{6}} = \frac{n}{2} \cdot \frac{2n \,.\, 6}{2n(4n^2-1)} = \frac{3n}{(4n^2-1)}$$

***Example* 5:**

**If the letters of the word REGULATIONS be arranged such that only *R* and *E* can change places and the rest have same order among themselves, what is the chance that there are exactly four letters between *R* and *E*?**

***Solution:***

There are 11 places corresponding to 11 letters and in these 11 places

*R* and *E* can be arranged in and $^{11}P_2$ ways.

With regard to favourable case

*R x x x x E* … *R* is in the first place from the left and *E* in the sixth place

*X R x x x x E* … *R* is in the second place and *E* in the 7th place.

*x x x x x R x x x x E* and thus *R* is 6th and *E* is 11th; besides *R* and *E* can

be interchanged also. Probability $= \frac{2 \times 6}{^{11}P_2} = \frac{2 \times 6}{11.10} = \frac{6}{55}$

**Note:** The working is based on the relative positions of *R* and *E*. Other letters do not play any part.

***Example* 6:**

**Cards are dealt one by one from a well shuffled pack until an ace appears if cards are not replaced. Find the probability that exactly *n* cards are dealt before the first appears.**

***Solution:***

The question envisages that in the first *n* draws there is no ace at all. Hence the probability of this happening $= \frac{^{48}C_n}{^{52}C_n}$.

The $(n + 1)$ th draw is an ace, for which the probability is $\frac{4}{52 - n}$

Hence the required probability $= \frac{^{48}C_n}{^{52}C_n} \cdot \frac{4}{52 - n}$

$$= \frac{\lfloor 48 \, \lfloor n \, \lfloor 52 - n}{\lfloor n \, \lfloor 48 - n \, \lfloor 52} \cdot \frac{4}{52 - n} = \frac{4 \,.\, (49 - n)\,(50 - n)\,(51 - n)}{49 \,.\, 50 \,.\, 51 \,.\, 52}$$

***Example* 7:**

**The probabilities that a student passes in Mathematics, Physics and Chemistry are in m, *p* and c, respectively. Of these subjects, the student has 75% chance of passing in at least one, 50% chance of passing in at least two, and a 40% chance of passing in exactly two. Then find the relation amount p, m and c.**

***Solution*:**

We have, $P(A \cup B \cup C) = \frac{3}{4}$ ...(i)

i.*e*. $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) = \frac{3}{4}$

$$P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C) = \frac{1}{2} \qquad \text{...(ii)}$$

and $P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C) = \frac{2}{5}$ ...(iii)

From (ii) and (iii), we get $P(A \cap B \cap C) = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$ ...(iv)

$\Rightarrow$ $P(A)P(B)P(C) = \frac{1}{10}$ $\Rightarrow$ $pmc = \frac{1}{10}$

From (i), (ii) and (iii), we have

$$P(A) + P(B) + P(C) - \left(\frac{1}{2} + \frac{2}{10}\right) + \frac{1}{10} = \frac{3}{4} \Rightarrow p + m + c = \frac{27}{20}$$

$\therefore$ Either $pmc = \frac{1}{10}$ or $p + m + c = \frac{27}{20}$ is the desired relation.

***Example* 8:**

**In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random find the probability that**

**(i) the student opted for NCC or NSS.**

**(ii) the student has opted neither NCC nor NSS.**

**(iii) the student has opted NSS but not NCC.**

***Solution:***

Let $E_1$ : student opted for NCC

$E_2$ : student opted for NSS,

then $P(E_1) = \frac{30}{60} = \frac{1}{2}$, $P(E_2) = \frac{32}{60} = \frac{8}{15}$ and $P(E_1 \text{ and } E_2) = \frac{24}{60} = \frac{2}{5}$

(i) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2) = \frac{1}{2} + \frac{8}{15} - \frac{2}{5} = \frac{15 + 16 - 12}{30} = \frac{19}{30}$

$\therefore$ *P*(student opted NCC or NSS) = $\frac{19}{30}$

(ii) $P$(student opted neither NCC nor NSS)

$$= P\left(E_1' \text{ and } E_2'\right)$$

$$= P\left(E_1' \cap E_2'\right) = P\left(\left(E_1 \cup E_2\right)'\right)$$

$$= 1 - P\left(E_1 \cup E_2\right)$$

$$= 1 - \frac{19}{30} \qquad \{\text{using part (i)}\}$$

$$= \frac{11}{30}$$

(iii) $P$(student has opted NSS but not NCC)

$$= P(E_2 \text{ but not } E_1)$$

$$= P\left(E_2 \cap E_1'\right)$$

$$= P\left(E_2\right) - P\left(E_2 \cap E_1\right)$$

$$= \frac{8}{15} - \frac{2}{5} = \frac{8-6}{15} = \frac{2}{15}$$

# EXERCISE – I

**1.** A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.

**2.** If from a pack of 52 cards two are drawn at random, what is the probability that they will be both spade, heart or club.

**3.** A dice is rolled twice. Describe the following events associated with this experiment:

$E_1$ : "getting a total of 7 on the two throws"

$E_2$ : "getting an odd number on the first throw and even number on the second throw"

$E_3$ : "getting an odd number on first throw and a total of 9"

$E_4$ : "getting an odd number on first throw and a multiple of 3 on the second throw"

$E_5$ : "getting a total of more than 9 on the two throws".

**4.** A book contains 100 pages. A page is chosen at random. What is the chance that the sum of the digits on the page is equal to 9.

**5.** In an entrance test, that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing at least one of them is 0.95. What is the probability of passing both?

**6.** Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.

**7.** $A$ and $B$ are two events such that $P(A) = 0.54$, $P(B) = 0.69$ and $P(A \cap B) = 0.35$. Find

(i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$

**8.** If 4-digits numbers greater than 5000 are randomly formed from the digits 0, 1, 3, 5 and 7 what is the probability of forming a number divisible by 5 when,

(i) the digits are repeated?

(ii) the repetition of digits is not allowed?

**9.** The probability that a student will pass in Mathematics is $\frac{3}{5}$ and the probability that he will pass in English is $\frac{1}{3}$. If the probability that he will pass in both Mathematics and English is $\frac{1}{8}$, what is the probability that he will pass in at least one subject?

**10.** A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number formed is divisible by 4.

# EXERCISE – II

**1.** A bag contains 7 black and 4 white balls. Two balls are drawn simultaneously from the bag. Find the probability that
(i) one is white and the other is black
(ii) at least one white ball is selected

**2.** In the random arrangement of the letters of the word "BANANA", find the probability that
(i) all vowels come together.
(ii) all vowels are not together.

**3.** In the random arrangement of the letters of the word "DIRECTOR" what is the probability that
(i) vowels are together?
(ii) vowels are never together?

**4.** If six boys and six girls sit in a row randomly. Find the probability that
(i) all the girls are together.
(ii) all the girls are never together.
(iii) all the girls sit together but either side of the corner.
(iv) boys and girls sit alternately.
(v) a particular boy $B_1$ sits with a particular girl $G_1$.

**5.** An antiaircraft gun can take a maximum of four shots at an enemy plane moving away from itself. The probability of hitting the plane at the 1st, 2nd, 3rd, and 4th shot is 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that
(i) Gun will not hit the plane?
(ii) Gun will target the plane?

**6.** The probability of *A, B, C* solving a problem are $\frac{1}{3}, \frac{2}{7}, \frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, the probability that exactly one of them will solve it.

**7.** In a college 25% of boys and 10% of girls are offered mathematics. The girls constitute 60% of the total number of students. If a student is selected randomly and is found to be studying mathematics, then find the probability that the student is a girl.

**8.** The probability that A speaks truth is $\frac{4}{5}$, while this for $B$ is $\frac{3}{4}$, the probability that they contradict each other when asked to speak on a fact.

**9.** A card is drawn randomly from a pack of playing cards then find the probability by two different approaches to verify your answer that it is neither ace nor king.

**10.** The probability of getting a doublet with 2 dice is.

# ANSWERS

## ANSWERS TO PRACTICE PROBLEMS

**PP1.** $S = \{(X, B_1), (X, B_2), (X, G_1), (X, G_2), (Y, b_1), (Y, g_1), (Y, g_2), (Y, g_3)\}$

**PP2.** $S = \left\{\begin{matrix} 6, (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (1, 1, 6), (1, 2, 6), (1, 3, 6), \\ (1, 4, 6), (1, 5, 6), (2, 1, 6), (2, 2, 6), ...... \end{matrix}\right\}$

**PP3.** No

**PP4.** (i) $\frac{1}{12}$ (ii) $\frac{1}{12}$

**PP5.** $\frac{1}{38760}$

**PP6.** $\frac{2}{7}$

**PP7.** $\frac{1}{132}$

**PP8.** $\frac{23}{24}$

**PP9.** $\frac{5}{9}$

**PP10.** $\frac{125}{168}$

# ANSWERS TO EXERCISE – I

**1.** $S = \{RW, WR, WW\}$

**2.** $\frac{3}{17}$

**3.** $E_1$ : {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}
$E_2$ : {(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)}
$E_3$ : {(3, 6), (5, 4)}
$E_4$ : {(1, 3), (1, 6), (3, 3), (3, 6), (5, 3), (5, 6)}
$E_5$ : {(4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)}

**4.** $\frac{1}{10}$

**5.** 0.55

**6.** $\frac{2}{3}$

**7.** (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34

**8.** (i) $\frac{2}{5}$ (ii) $\frac{3}{8}$

**9.** $\frac{97}{120}$

**10.** $\frac{1}{5}$

# ANSWERS TO EXERCISE – II

**1.** (i) 0.509 (ii) 0.618

**2.** (i) $\frac{1}{5}$ (ii) $\frac{4}{5}$

**3.** (i) $\frac{3}{28}$ (ii) $\frac{25}{28}$

**4.** (i) $\frac{1}{132}$ (ii) $\frac{131}{132}$ (iii) $\frac{2 \times 6! \times 6!}{12!}$ (iv) $\frac{2 \times 6! \times 6!}{12!}$ (v) $\frac{1}{6}$

**5.** (i) 0.3024 (ii) 0.6976

**6.** $\frac{25}{56}$

**7.** $\frac{3}{8}$

**8.** $\frac{7}{20}$

**9.** $\frac{11}{13}$

**10.** $\frac{1}{6}$