LESSON 12

THREE DIMENSIONAL GEOMETRY

1. INTRODUCTION

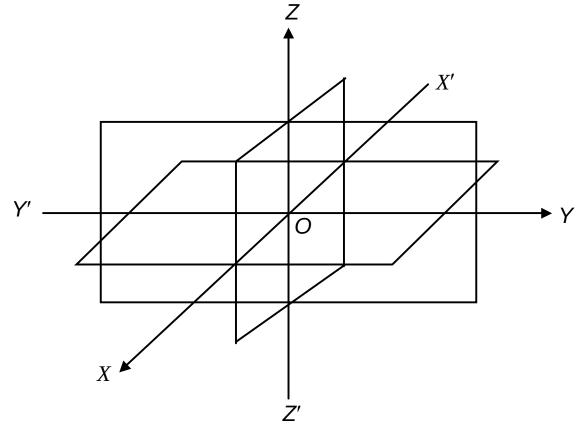
To locate a point in a plane we need a two dimensional coordinate system. On similar lines to locate a point in space we use a three dimensional coordinate system.

Consider three planes intersecting at a point *O*. The coordinate axes *x*-axis, y-axis and z-axis are shown by dark lines.

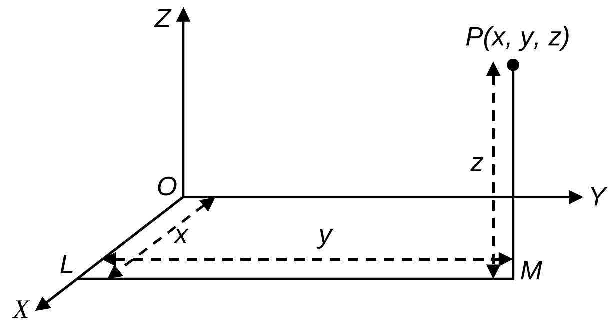
i.e., if we have an insect on the floor of a room, we can specify its position by a two dimensional coordinate system but if a spider is hanging from the roof of a room we specify its position in a three dimensional coordinate system.

2. COORDINATE AXES

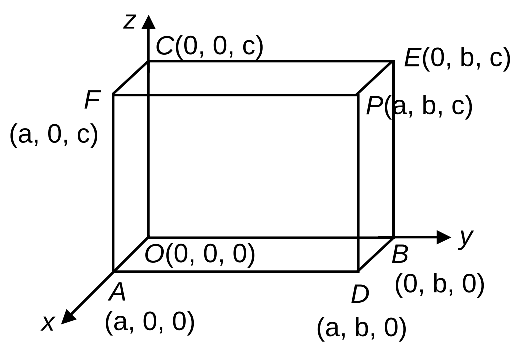
Three dimensional coordinate system is formed by taking three mutually perpendicular planes intersecting at a point O.



The lines XOX′, YOY′ and ZOZ′ at which the three planes intersect are mutually perpendicular to each other and are referred to as the *x*-axis, y-axis and z-axis respectively. The distance measured from XY plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. We specify the coordinates of a point by specifying the *x*, *y* and *z* coordinates of a point.



If the coordinates of *P* are (*a*, *b*, *c*), we mark the coordinates of other points as shown in the figure.



3. DISTANCE BETWEEN TWO POINTS

To find the distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space.

From the above figure we note that,

$\angle PAN = 90°$

Applying

Pythagoras theorem,

$PN^2 = PA^2 + AN^2$

$\angle PNQ = 90°$

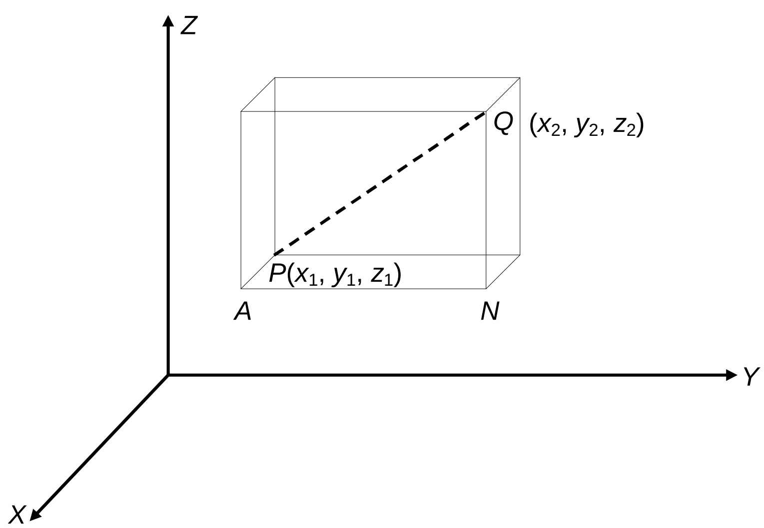
$\therefore \; PQ^2 = PN^2 + NQ^2$

$\therefore \; PQ^2 = PA^2 + AN^2 + NQ^2$

$PA = (x_2 - x_1)$

$AN = (y_2 - y_1)$

$NQ = (z_2 - z_1)$



$\therefore \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$\therefore \qquad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Illustration 1

Question: **Find the distance between the of points *A*(2, J1, 3) and *B*(J2, J1, 3).**

Solution: Distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$AB = \sqrt{(2+2)^2 + (-1+1)^2 + (3-3)^2}$ = 4 units

Illustration 2

Question: **Prove by using distance formula that the points *P*(1, 2, 3), *Q*(J1, J1, J1) and *R*(3, 5, 7) are collinear.**

Solution: We have $PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2}$

$= \sqrt{4+9+16} = \sqrt{29}$

$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2}$

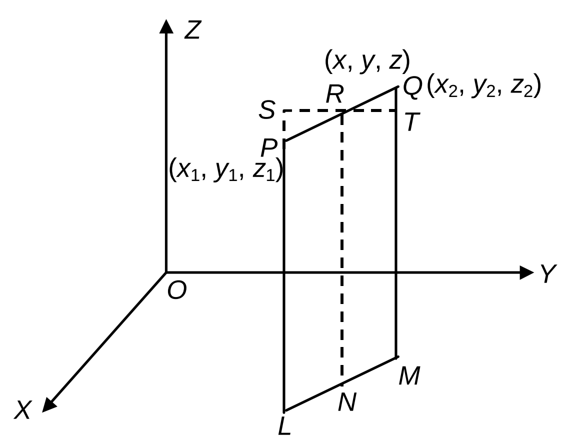
$= \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$

and $PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since $QR = PQ + PR$. Therefore the given points are collinear.

4. SECTION FORMULA

To find the coordinates of a point R which divides PQ internally in the ratio $m : n$, drop perpendicular from P, R and Q on the xy plane. Let the foot of perpendiculars on the xy plane be denoted as L, N and M.



$\because$ ΔPRS is similar to ΔRQT.

$\therefore$ the corresponding sides are proportional.

$$\therefore \qquad \frac{PR}{RQ} = \frac{m}{n} = \frac{SP}{QT} = \frac{SR}{RT}$$

Now, $SP = (SL - PL) = (RN - PL) = (z - z_1)$

$$QT = (QM - TM) = (QM - RN) = (z_2 - z)$$

$\therefore \quad \dfrac{m}{n} = \dfrac{(z - z_1)}{(z_2 - z)}$

$\therefore \quad (mz_2 - mz) = nz - nz_1 \quad \Rightarrow \quad z(m + n) = mz_2 + nz_1$

$$z = \frac{(mz_2 + nz_1)}{(m + n)}$$

Similarly,

$$y = \frac{(my_2 + ny_1)}{(m + n)}, \quad x = \frac{(mx_2 + nx_1)}{(m + n)}$$

$\therefore$ The coordinates of point R are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

Similarly if R divides PQ externally in the ratio of $m : n$, then the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

Illustration 3

Question: **Find the ratio in which *yz*–plane divides the line joining (2, 4, 5) and (3, 5, 7).**

Solution: Let the ratio be $\lambda : 1$

x-coordinate is $\dfrac{3\lambda + 2}{\lambda + 1} = 0 \quad \Rightarrow \lambda = -\dfrac{2}{3}$

$\therefore$ 2 : 3 (externally)

Illustration 4

Question: **Show that the points (2, –3, 4), (–1, 2, 1) and $\left(0, \dfrac{1}{3}, 2\right)$ are collinear and find the ratio in which the third point divides the line joining first two.**

Solution: Let A denote the point (2, –3, 4) and B the point (–1, 2, 1)

Let C divides the line joining A and B in the ratio of $\lambda : 1$

$\therefore$ co-ordinates of C are

$$\left(\frac{2 - \lambda}{\lambda + 1}, \frac{-3 + 2\lambda}{\lambda + 1}, \frac{4 + \lambda}{\lambda + 1}\right) \equiv \left(0, \frac{1}{3}, 2\right) \text{ \{given\}}$$

$\therefore \quad \dfrac{2 - \lambda}{\lambda + 1} = 0 \quad \Rightarrow \quad \lambda = 2$

and $\dfrac{-3+2\lambda}{\lambda+1}=\dfrac{1}{3}$

$\Rightarrow$ $-9 + 6\lambda = \lambda + 1$

$\Rightarrow$ $5\lambda = 10$ $\Rightarrow$ $\lambda = 2$

Also $\dfrac{4+\lambda}{\lambda+1}=2$

$\Rightarrow$ $4 + \lambda = 2\lambda + 2$

$\Rightarrow$ $\lambda = 2$

From each equation we get the value of λ as 2.

$\therefore$ The point *A*, *C* and *B* are collinear.

$\therefore$ The point *C* divides the line joining *A* and *B* internally in the ratio of 2 : 1.

PRACTICE PROBLEMS

PP1. Show that the points (a, b, c), (b, c, a) and (c, a, b) are the vertices of an equilateral triangle.

PP2. Find the ratio in which the point C(5, 9, –14) divides the join of A(2, –3, 4), B(3, 1, –2).

PP3. Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2 : 3 (i) internally and (ii) externally.

PP4. Find the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

PP5. Find the distance of the point $P(a, b, c)$ from x-axis.

PP6. Find the area of the triangle formed by joining the points whose coordinates are (1, 0, 0), (1, 2, 0) and (1, 0, 3) in sq. units.

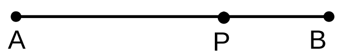
SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find the co-ordinate of the point *P* which is five sixth of the way from *A*(–2, 0, 6) to *B*(10, –6, –12).

Solution :

Let $P(x,\ y,\ z)$ be the required point



$\Rightarrow$ *P* divides *AB* in ratio 5 : 1

$$\left[\because\ AP = \frac{5}{6}AB \Rightarrow \frac{AP}{AB} = \frac{5}{6} \Rightarrow \frac{AP}{AB - AP} = \frac{5}{6-5} \Rightarrow \frac{AP}{PB} = \frac{5}{1}\right]$$

$$\Rightarrow (x,\ y,\ z) = \left(\frac{5\times 10 + 1\times -2}{5+1},\ \frac{5\times -6 + 1\times 0}{5+1},\ \frac{5\times -12 + 1\times 6}{5+1}\right)$$

$\Rightarrow (x,\ y,\ z) = (8,\ -5,\ -9) \Rightarrow P(8,\ -5,\ -9)$ is the required point.

***Example* 2:**

Prove that the line through $A(0, -1, -1)$, $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$.

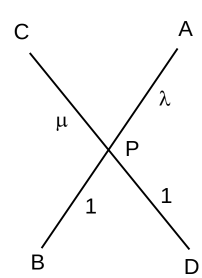
Solution :

The lines *AB* and *CD* intersect if there exists a common point *P* dividing *AB* in some ratio say $\lambda : 1$ and *CD* in some ratio say $\mu : 1$

where $\lambda + 1 \neq 0,\ \mu + 1 \neq 0$

i.e., if $\left(\frac{4\lambda}{\lambda+1}, \frac{5\lambda-1}{\lambda+1}, \frac{\lambda-1}{\lambda+1}\right) = \left(\frac{-4\mu+3}{\mu+1}, \frac{4\mu+9}{\mu+1}, \frac{4\mu+4}{\mu+1}\right)$

$$\text{if } \begin{cases} \dfrac{4\lambda}{\lambda+1} = \dfrac{-4\mu+3}{\mu+1} \\ \dfrac{5\lambda+1}{\lambda+1} = \dfrac{4\mu+9}{\mu+1} \\ \dfrac{\lambda-1}{\lambda+1} = 4 \text{ i.e., } \lambda = \dfrac{-5}{3} \end{cases} \text{ are consistent.}$$



i.e., if $\dfrac{4\left(-\frac{5}{3}\right)}{\frac{-5}{3}+1} = \dfrac{-4\mu+3}{\mu+1}, \dfrac{5\left(-\frac{5}{3}\right)-1}{\frac{-5}{3}+1} = \dfrac{4\mu+9}{\mu+1}, \lambda = -\dfrac{-5}{3}$ are consistent.

i.e., if $\dfrac{-20}{-2} = \dfrac{-4\mu+3}{\mu+1}, \dfrac{-28}{-2} = \dfrac{4\mu+9}{\mu+1}, \lambda = -\dfrac{-5}{3}$ are consistent.

i.e., if $10 = \dfrac{-4\mu+3}{\mu+1}, 14 = \dfrac{4\mu+9}{\mu+1}, \lambda = -\dfrac{-5}{3}$ are consistent.

i.e., if $14\mu = -7, 10\mu = -5, \lambda = \dfrac{-5}{3}$ are consistent.

i.e., if $\mu = -\dfrac{1}{2}, \mu = -\dfrac{1}{2}, \lambda = \dfrac{-5}{3}$ which is true.

$\therefore$ Lines *AB* and *CD* intersect in *P*.

***Example* 3:**

Using section formula, prove that the three points $(-2, 3, 5)$, $(1, 2, 3)$ and $(7, 0, -1)$ are collinear. Also find the ratio in which the third point divides the segment joining first two points.

Solution :

Let $A(-2, 3, 5), B(1, 2, 3), C(7, 0, -1)$ be the given points.

The point which divides *AB* in the ratio $k : 1$ is $\left(\dfrac{k-2}{k+1}, \dfrac{2k+3}{k+1}, \dfrac{3k+5}{k+1}\right)$ …(i)

If this is the point $C(7, 0, -1)$, then $\dfrac{k-2}{k+1} = 7, \dfrac{2k+3}{k+1} = 0, \dfrac{3k+5}{k+1} = -1$

First equation gives $k - 2 = 7k + 7$

i.e., $6k = -9$, $\therefore \quad k = -\dfrac{9}{6} = -\dfrac{3}{2}$

Second equation gives $2k + 3 = 0, \therefore k = -\dfrac{3}{2}$

Third equation gives $3k + 5 = -k - 1$ or $4k = -6$

$\therefore \quad k = -\dfrac{6}{4} = \dfrac{-3}{2}$

Hence for $k = -\dfrac{3}{2}$ the point given (i) coincides with *C*.

$\therefore$ *C* divides *AB* externally in the ratio 3 : 2

Thus *A, B, C* are collinear.

***Example* 4:**

Find the ratio in which the surface $x^2 + y^2 + z^2 = 25$ divides the line segment joining (0, 1, 2) and (3, 4, 5).

Solution :

Given surface is $x^2 + y^2 + z^2 = 25$

Let $A \equiv (0, 1, 2), B \equiv (3, 4, 5)$ ...(i)

Let surface (i) divide the line segment *AB* internally in the ratio *k* : 1 at *P*.

Then $P = \left(\frac{3k}{k+1}, \frac{4k+1}{k+1}, \frac{5k+2}{k+1} \right)$

Since *P* lies on surface (i),

$\therefore \quad \frac{9k^2 + (4k+1)^2 + (5k+2)^2}{(k+1)^2} = 25$ or $50k^2 + 28k + 5 = 25(k^2 + 2k + 1)$

or $25k^2 - 23k - 20 = 0$ or $k = \frac{23 \pm \sqrt{2599}}{50} = \frac{23 + \sqrt{2529}}{50}, \frac{23 - \sqrt{2529}}{50}$

Hence surface (i) divides the line segment *AB* internally in the ratio $\frac{23 + \sqrt{2529}}{50}$ and externally in the ratio $\frac{23 - \sqrt{2599}}{50}$.

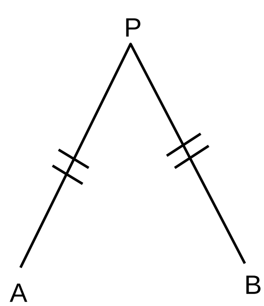
***Example* 5:**

Find the locus of a locus of a point while moves so that its distances from the points (3, 4, –5) and (–2, 1, 4).

Solution :

Let $A = (3, 4, -5), B = (-2, 1, 4)$

Let $P(x, y, z)$ be any point on the locus.



$\therefore$ By the given conditions. $PA = PB$

$\therefore \quad \sqrt{(x-3)^2 + (y-4)^2 + (z+5)^2} = \sqrt{(x+2)^2 + (y-1)^2 + (z-4)^2}$

$\Rightarrow \quad x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 + 10z$

$= x^2 + 4 + 4x + y^2 + 1 - 2y + zh2 + 16 - 8z$

$\Rightarrow \quad -10x - 6y + 18z = 29$

$\Rightarrow\ 10x + 6y - 18z + 29 = 0$

***Example* 6:**

Find the equation of the set of points *P*, the sum of whose distance from *A*(4, 0, 0) and *B*(–4, 0, 0) is equal to 10.

Solution :

Given points are $A(4, 0, 0)$ and $B(-4, 0, 0)$

Let $P(x, y, z)$ be any point such that $|PA| + |PB| = 10$

$$\Rightarrow\ \sqrt{(x-4)^2 + (y-0)^2 + (z-0)^2} + \sqrt{(x+4)^2 + (y-0)^2 + (z-0)^2} = 10$$

$$\Rightarrow\ \sqrt{(x+4)^2 + y^2 + z^2} = 10 - \sqrt{(x-4)^2 + y^2 + z^2}$$

Squaring the two sides, we get

$$(x+4)^2 + y^2 + z^2 = 100 + (x-4)^2 + y^2 + z^2 - 20\sqrt{(x-4)^2 + y^2 + z^2}$$

or $(x+4)^2 - (x-4)^2 - 100 = -20\sqrt{(x-4)^2 + y^2 + z^2}$

or $16x - 100 = -20\sqrt{(x-4)^2 + y^2 + z^2}$

Dividing by -4, we obtain

$$-4x + 25 = 5\sqrt{(x-4)^2 + y^2 + z^2}$$

Squaring, we get $16x^2 + 625 - 200x = 25\{x^2 - 8x + 16 + y^2 + z^2\}$

or $9xh2 + 25y^2 + 25z^2 = 225$ or $\dfrac{9x^2}{225} + \dfrac{25y^2}{225} + \dfrac{25z^2}{225} = 1$

or $\dfrac{x^2}{25} + \dfrac{y^2}{9} + \dfrac{z^2}{9} = 1$, which is the required equation.

EXERCISE – I

1. Name the octants in which the following points lie:
(1, 2, 3), (4, –2, –5), (–3, 2, –5) and (–4, –6, 7).

2. Find the distance between the points *A* (5, 1, 2) and *B* (4, 6, –1).

3. Show that *A*(–2, 3, 5), *B*(1, 2, 3) and *C*(7, 0, –1) are collinear.

4. Determine the values of *a* and *b* so that the points $(a,\ b,\ 3)$, $(2,\ 0,\ -1)$ and $(1,\ -1,\ -3)$ are collinear.

5. Let *A*(2, 1, –3) and *B*(5, –8, 3) be two given points. Find the coordinates of the points of trisection of the line segment *AB*.

6. Show that the points *A*(1, 1, 1), *B*(–2, 4, 1), *C*(–1, 5, 5) and *D*(2, 2, 5) are the vertices of a square.

7. Find the coordinates of the point where the line joining *A*(3, 4, 1) and *B*(5, 1, 6) crosses the *xy*-plane.

8. If the three consecutive vertices of a parallelogram be *A*(3, 4, –1), *B*(7, 10, –3) and *C*(5, –2, 7), find the fourth vertex *D*.

9. The midpoints of the sides of a triangle are (1, 5, –1), (0, 4, –2) and (2, 3, 4). Find its vertices.

10. Find the point in *yz*-plane which is equidistant from the points *A*(3, 2, –1), *B*(1, –1, 0) and *C*(2, 1, 2).

EXERCISE – II

1. Planes are drawn parallel to the co-ordinate planes through the points $(1, 2, 3)$ and $(3, -4, -5)$. Find the length of diagonal of parallelepiped so formed.

2. Show that the line segment joining the points $A(1, 2, 3)$, $B(4, 5, 6)$ is parallel to the line segment joining the points $C(-4, 3, -6)$ and $D(2, 9, 0)$.

3. Find the length of medians of the triangle having vertices $A(0, 0, 6)$, $B(0, 4, 0)$ and $C(6, 0, 0)$.

4. Find the coordinates of a point on y-axis which are at a distance of $5\sqrt{2}$ from the point $P(3, -2, 5)$.

5. The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, –6) respectively, find the coordinates of the point C.

6. Let A, B, C be the feet of perpendiculars drawn from a point P to x, y and z axes respectively. Find the co-ordinates of A, B, C; if co-ordinates of P are

(i) (3, 1, 2) (ii) (3, –6, 2)

7. Find the equation of the set of points P, the sum of whose distances from $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10.

8. Find the locus of a point P which moves in such a way that $2PA = 3PB$, where $A(-2, 1, 3)$, and $B(3, -1, 2)$ are the given points.

9. Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7) respectively.

10. The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are $(3, -5, 7)$ and $(-1, 7, -6)$ respectively, find the coordinates of the point C.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP2. (3 : 2 externally)

PP3. (–3, –14, 19)

PP4. $\left(\frac{\sum x_1}{3}, \frac{\sum y_1}{3}, \frac{\sum z_1}{3}\right)$

PP5. $\sqrt{b^2 + c^2}$

PP6. 3

ANSWERS TO EXERCISE – I

1. I, VIII, VI, III

2. $\sqrt{35}$ units

4. $a = 4$, $b = 2$

5. (3, –2, –1), (4, –5, 1)

7. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

8. (1, –8, 9)

9. (1, 2, 3) (3, 4, 5) (–1, 6, –7)

10. $\left(0, \frac{31}{16}, \frac{-3}{16}\right)$

ANSWERS TO EXERCISE – II

1. $\sqrt{104}$

3. 7, $\sqrt{34}$, 7

4. (0, 2, 0) and (0, –6, 0)

5. (1, 1, 2)

6. (i) $A(3, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 2)$
(ii) $A(3, 0, 0)$, $B(0, -6, 0)$, $C(0, 0, 2)$

7. $9x^2 + 25y^2 + 25z^2 = 225$

8. $5(x^2 + y^2 + z^2) - 70x + 26y - 12z + 70 = 0$

9. $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$

10. (1, 1, 2)