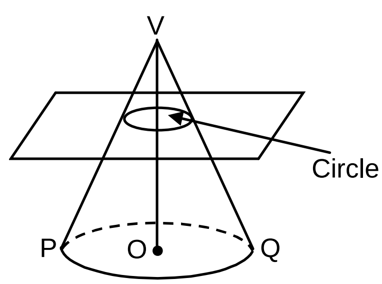
LESSON 11

CONIC SECTION

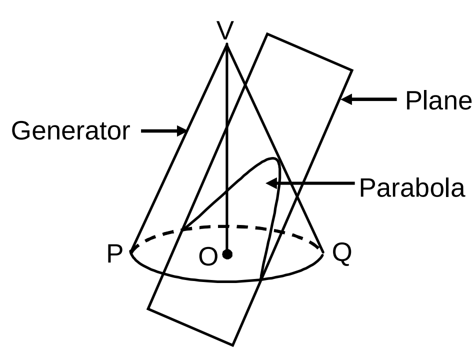
Take two sticks with one end common and different other end. Keep one stick fixed and rotate the other around it by 360°. The figure so generated is a cone. The fixed stick is the axis and the one moved around is referred to as the generator of the cone.

And when a plane cuts the cone at different angles we get different sections of the cone namely circle, parabola, ellipse and hyperbola.

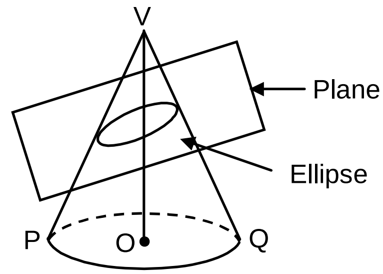
Now if a plane cuts the cone such that it is perpendicular to the axis of the cone, the section is a circle.



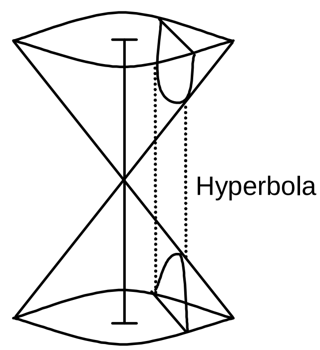
Section of a right circular cone cut by a plane, which is parallel to a generator of the cone is a parabola.



Section of a right circular cone by a plane which is not parallel to any generator and not parallel or perpendicular to the axis of the cone is an ellipse.



Section of a right circular cone by a plane which is parallel to the axis of the cone is a hyperbola.



In the following sections we shall obtain the equations of each of the conic sections in standard form.

1. CIRCLE

A circle is the set of all points in a plane which are at a constant distance from a fixed point in the plane.

OR

A circle is defined as the locus of a point which moves in a plane such that it remains at a constant distance from a fixed point in that plane.

The fixed point is called the **centre** and the constant distance called the **radius** of the circle.

1.1 EQUATION OF CIRCLE IN VARIOUS FORMS

- **Equation of circle with centre $C(\alpha, \beta)$ and radius r.**

Let $P(x, y)$ be any point on the circle.

Then, by definition $|CP| = r$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = r$$

$$(x - \alpha)^2 + (y - \beta)^2 = r^2 \qquad \text{...(i)}$$

Equation (i) gives us the required equation of the circle with centre (α, β) and radius r.

- **Equation of circle with centre origin *O*(0, 0) and radius *r*.**

  Substituting $\alpha = 0$ and $\beta = 0$ in (i), we have

  $$x^2 + y^2 = r^2 \qquad \text{...(ii)}$$

  Equation (ii) gives us the required equation of the circle.

- **General Equation of a circle**

  General equation of second degree in x and y is

  $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(iii)}$$

  The equation of a circle with centre (α, β) and radius r is

  $$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

  $$x^2 + y^2 - 2\theta x - 2\beta y + (\alpha^2 + \beta^2 - r^2) = 0 \qquad \text{...(iv)}$$

  Comparing (iii) and (iv)

  $\Rightarrow$ $a = b$ i.e. coefficient of x^2 = coefficient of y^2.

  $h = 0$ i.e. coefficient of $xy = 0$.

  $\therefore$ the general equation of a circle is

  $$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(v)}$$

  Comparing (iv) and (v),

  $$\alpha = -g; \qquad \beta = -f; \qquad r = \sqrt{g^2 + f^2 - c}$$

  $\therefore$ centre of the circle is $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

Illustration 1

Question: **Find the equation of the circle with centre ($a\cos\theta$, $a\sin\theta$) and radius a.**

Solution: The standard equation of the circle is $(x - \alpha)^2 + (y - \beta)^2 = r^2$

$\therefore$ $(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = r^2$

$x^2 + a^2\cos^2\alpha - 2ax\cos\alpha + y^2 + a^2\sin^2\alpha - 2ay\sin\alpha = r^2$

$x^2 + y^2 + a^2 - 2ax\cos\alpha - 2ay\sin\alpha = r^2$

$x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha + (a^2 - r^2) = 0.$

Illustration 2

Question: **Find the equation of the circle passing through the points (1, −2), (5, 4) and (10, 5).**

Solution: The general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Points (1, −2), (5, 4), (10, 5) satisfy equation (i)

$\therefore 1 + 4 + 2g - 4f + c = 0$

$$2g - 4f + c + 5 = 0 \qquad \text{...(ii)}$$

$25 + 16 + 10g + 8f + c = 0$

$10g + 8f + c + 41 = 0$ ...(iii)

$100 + 25 + 20g + 10f + c = 0$

$20g + 10f + c + 125 = 0$ ...(iv)

Solving equations (ii), (iii) and (iv) we have,

$g = -9$; $f = 3$; $c = 25$

substituting the values of *f, g* and *c* in equation (i) $x^2 + y^2 - 18x + 6y + 25 = 0$.

Illustration 3

Question: **Find the equation of the circle passing through the point *P*(2, 4) and centre at the intersection of the lines $x - y = 4$ and $2x + 3y = -7$.**

Solution: Solving the equation $x - y = 4$ and $2x + 3y = -7$, we have point of intersection of these lines as *C*(1, –3)

$$CP = \sqrt{(2-1)^2 + (4+3)^2} = \sqrt{50}$$

Now equation of circle with centre *C*(1, –3) and radius *CP* is given by

$$(x-1)^2 + (y+3)^2 = (\sqrt{50})^2$$

$$x^2 + y^2 - 2x + 6y - 40 = 0$$

Illustration 4

Question: **Find the equation of circle of radius 5, whose centre lies on *y*-axis and passes through *P*(3, 2).**

Solution: Any point on *y*-axis can be taken as *A*(0, *k*) according to question, $AP = 5$

$$\sqrt{(3-0)^2 + (2-k)^2} = 5 \qquad k^2 - 4k - 12 = 0 \qquad k = -2, 6$$

Now, the equation of circle with centre (0, –2) and radius 5 is $(x-0)^2 + (y+2)^2 = 25$

$$x^2 + y^2 + 4y - 21 = 0 \qquad \text{...(i)}$$

Also, the equation of circle with centre (0, 6) and radius 5 is $(x-0)^2 + (y-6)^2 = 25$

$$x^2 + y^2 - 12y + 11 = 0 \qquad \text{...(ii)}$$

Equation (i) and (ii) gives the required circle.

Illustration 5

***Question*:** **For what value(s) of λ, the equation $(10 - \lambda^2)x^2 + (\lambda^2 - 8)y^2 + (3 - \lambda)yx - 10x + 4y + 3 = 0$ represent a real circle.**

Solution: We must have

coeff. of x^2 = coeff. of y^2 $\quad 10 - \lambda^2 = \lambda^2 - 8 \quad \lambda^2 = 9 \quad \lambda = \pm 3$

and coeff. of $xy = 0$ $\quad 3 - \lambda = 0 \quad \lambda = 3$

So, the common value is ⇐= 3.

Mathematically, a parabola, ellipse and hyperbola are expressed in terms of a fixed point called focus and a fixed line called directrix.

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant, is known as a conic section or a conic.

The fixed point is called the **focus** of the conic and the fixed line is called the **directrix** of the conic. Also this constant ratio is called the **eccentricity** of the conic and is denoted by e.

If $e = 1$, the conic is called parabola.

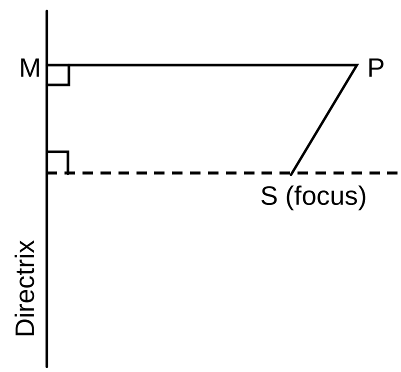
If $e < 1$, the conic is called ellipse.

If $e > 1$, the conic is called hyperbola.

In the figure

$$\frac{SP}{PM} = \text{constant} = e$$

or $\quad SP = e\,PM$

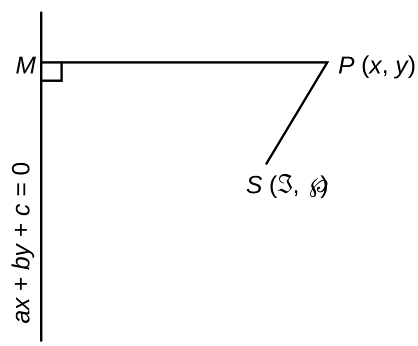


EQUATION OF CONIC SECTION (PARABOLA, ELLIPSE, HYPERBOLA)

If the focus is (α, β) and the directrix is $ax + by + c = 0$ then the equation of the conic section whose eccentricity $= e$ is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e.\frac{|ax+by+c|}{\sqrt{(a^2+b^2)}}$$

or $$(x-\alpha)^2 + (y-\beta)^2 = e^2.\frac{(ax+by+c)^2}{(a^2+b^2)}$$



IMPORTANT TERMS

Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex: The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal chord: Any chord passing through the focus is called focal chord of the conic section.

Double ordinate: A straight line drawn perpendicular to the axis and terminated at both ends of the curve is a double ordinate of the conic section.

Latus rectum: The double ordinate passing through the focus is called the latus rectum of the conic section.

Centre: The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

Let $F(x, y) = 0$ be the equation of a curve C, then C is symmetrical about

(a) X-axis if $F(x, y) = F(x, -y)$

(b) Y-axis if $F(x, y) = F(-x, y)$

(c) line $x - y = 0$ if $F(x, y) = F(y, x)$

(d) line $x + y = 0$ if $F(x, y) = F(-y, -x)$

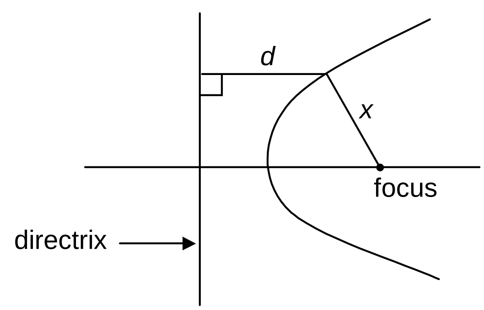
(e) origin if $F(x, y) = F(-x, -y)$

2. PARABOLA

2.1 DEFINITION

The locus of a point, which moves such that its distance from focus is equal to its perpendicular distance from the directrix, is a parabola.

i.e., $d = x$



2.2 EQUATION OF A PARABOLA IN THE STANDARD FORM

Let the coordinates of the vertices be (0, 0) and the distance of the directrix from the focus be $2a$. Let the coordinates of the focus be $(a, 0)$.

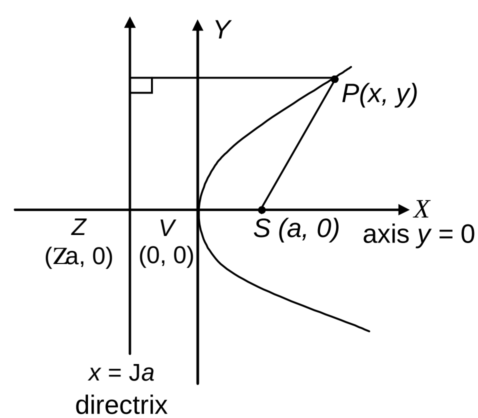
Take a point $P(x, y)$ on the parabola.

$\because$ from the definition of parabola,

$$\sqrt{(x-a)^2 + y^2} = |x + a|$$

$$(x - a)^2 + y^2 = (x + a)^2$$

$$y^2 = 4ax, \quad a > 0.$$



This is called the standard equation of parabola.

2.3 IMPORTANT TERMS

Axis: The axis of the parabola is the straight line which passes through focus and perpendicular to the directrix of the parabola.

For the parabola $y^2 = 4ax$, x-axis is the axis.

Here all powers of y are even in $y^2 = 4ax$ so parabola $y^2 = 4ax$ is symmetrical about its axis (i.e., x-axis).

Vertex: The point of intersection of the parabola and its axis is called the vertex of the parabola. For the parabola $y^2 = 4ax$, A (0, 0) is the vertex.

Latus rectum: The double ordinate LL′ passing through the focus is called the latus rectum of the parabola.

Since focus S (a, 0) the equation of the latus rectum of the parabola is $x = a$

solving $x = a$ and $y^2 = 4ax$

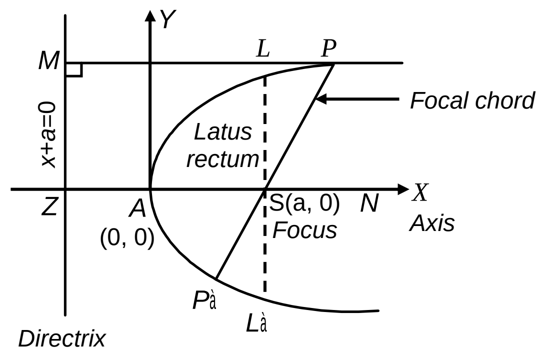
we get $y = \pm 2a$

Hence the coordinates of the extremities of the latus rectum are L $(a, 2a)$ and $L'(a, -2a)$ respectively.

Since $LS = L'S = 2a$

$\therefore$ Length of latus rectum

$LL' = 2\,(LS) = 2\,(L'S) = 4a$.



Focal chord: A chord of a parabola which passes through the focus is called a focal chord of the parabola.

In the given figure PP' and LL' are the focal chords.

Focal distance: The focal distance of any point P on the parabola is its distance from the focus S i.e., SP

Also $SP = PM$ = distance of P from the directrix

If $P \equiv (x, y)$; then $SP = PM = x + a$

Parametric equations: From the equation of the parabola $y^2 = 4ax$, we can write

$$\frac{y}{2a} = \frac{2x}{y} = t,$$ where t is a parameter. Then

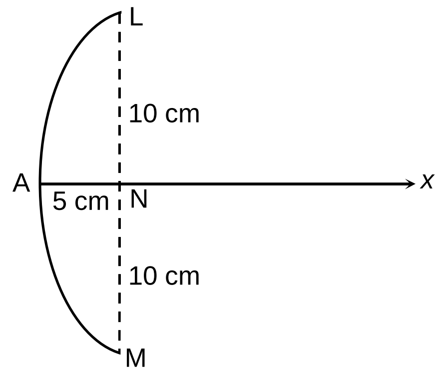
$y = 2at$ and $x = at^2$

The equations $x = at^2$ and $y = 2at$ are called parametric equations

Illustration 6

Question: **If a parabolic reflector is 20 cm in diameter and 5 cm deep, find its focus.**

***Solution*:** Let *LAM* be the parabolic reflector such that *LM* is its diameter and *AN* is its depth. It is given that *AN* = 5 cm and *LM* = 20 cm



∴ $LN = 10$ cm

Taking *A* as the origin, *AX* along *x*-axis and a line through *A* perpendicular to *AX* as *y*-axis, let the equation of the reflector be

$$y^2 = 4ax \qquad \ldots(i)$$

The point *L* has coordinates (5, 10) and lies on (i). Therefore $(10)^2 = 4a \times 5$ $\quad a = 5$

So, the equation of the reflector is $y^2 = 20x$

Its focus is at (5, 0) i.e. at point *N*

Hence, the focus is at the mid-point of the given diameter.

2.4 OTHER FORMS OF EQUATION OF A PARABOLA

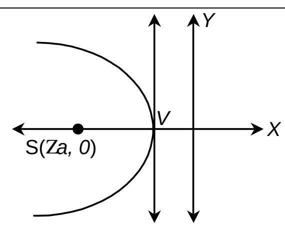
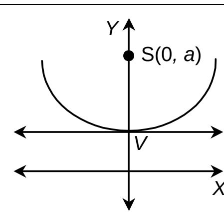
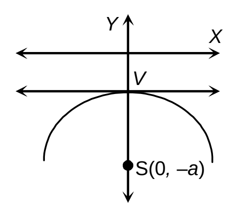
Equation	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graph			
Vertex	(0, 0)	(0, 0)	(0, 0)
Focus	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of directrix	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$x = 0$	$x = 0$
Equation of tangent at the vertex	$x = 0$	$y = 0$	$y = 0$
Latus rectum	$4a$	$4a$	$4a$
Extremities of latus rectum	$(-a, 2a)$ and $(-a, -2a)$	$(2a, a)$ and $(-2a, a)$	$(2a, -a)$ and $(-2a, -a)$
Parametric coordinates	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$

Illustration 7

Question: **For the parabola $y^2 = -12x$, find the coordinates of the focus and the equation of the directrix.**

***Solution*:** $y^2 = -12x$ ...(i)

Comparing equation (i) with the standard equation of the parabola $y^2 = 4ax$, we have

$4a = -12 \Rightarrow a = -3$; $\therefore$ focus $(a, 0) \equiv (-3, 0)$; directrix is $x = -a \Rightarrow x = 3$

Illustration 8

Question: **An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, whose vertex is at the vertex of the parabola. Find the length of its side.**

***Solution*:** Let $AB = l$, then

$AM = l\cos 30° = \dfrac{l\sqrt{3}}{2}$ and $BM = l\sin 30° = \dfrac{l}{2}$

So, the coordinates of B are $\left(\dfrac{l\sqrt{3}}{2}, \dfrac{l}{2}\right)$

Since, B lies on $y^2 = 4ax$

$\therefore \quad \dfrac{l^2}{4} = 4a\left(\dfrac{l\sqrt{3}}{2}\right) \Rightarrow l = 8a\sqrt{3}$

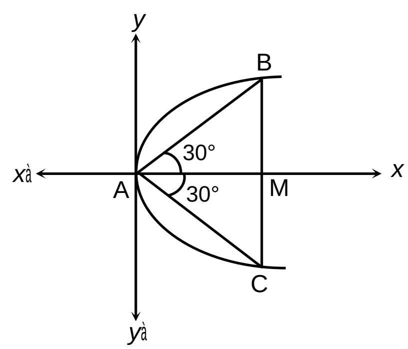


Illustration 9

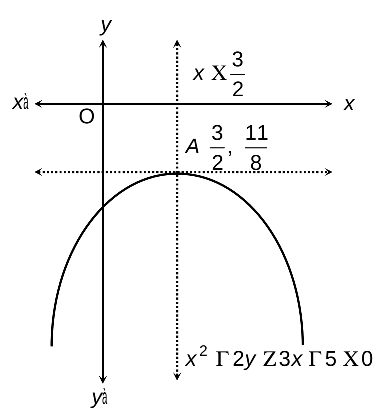
Question: **Find the vertex, axis, focus, directrix, latus–rectum of the parabola $x^2 + 2y - 3x + 5 = 0$. Also draw their rough sketch.**

***Solution*:** The given equation is $x^2 + 2y - 3x + 5 = 0$

$$x^2 - 3x = -2y - 5$$

$$x^2 - 3x + \frac{9}{4} = -2y - 5 + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right) \qquad \text{...(i)}$$



Shifting the origin to the point $(3/2, -11/8)$ without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we have

$$x = X + \frac{3}{2},\ y = Y - \frac{11}{8} \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$X^2 = -2Y \qquad \text{....(iii)}$$

This is of the form $X^2 = -4aY$. On comparing, we get $4a = 2$ i.e. $a = \dfrac{1}{2}$

Vertex: The coordinates of the vertex with respect to the new axes are (X = 0, Y = 0).

So, the coordinates to the vertex with respect to the old axes are

$\left(\frac{3}{2}, -\frac{11}{8}\right)$ [putting X = 0, Y = 0 in (ii)]

Axis: The equation of the axis of the parabola with respect to the new axes is X = 0.

So, the equation of the axis with respect to the old axes is

$x = \frac{3}{2}$ [putting X = 0 in (ii)]

Focus: The coordinates of the focus with respect to the new axes are (X = 0 , Y = –a) i.e. (X = 0, Y = –1/2).

So, the coordinates of the focus with respect to the old axes are

$\left(\frac{3}{2}, -\frac{15}{8}\right)$ [putting X = 0, Y = –1/2 in (ii)]

Directrix: The equation of the directrix with respect to the new axes is $Y = a$ i.e. Y = 1/2.

So, the equation of the directrix with respect to the old axes is

$y = -\frac{7}{8}$ [putting Y =1/2 in (ii)]

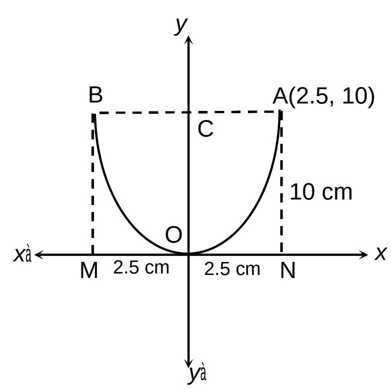
Latusrectum: The length of the latusrectum of the given parabola is equal to $4a = 2$.

Illustration 10

Question: **An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2m from the vertex of the parabola.**

***Solution*:** Let the vertex of the parabola be at the origin and axis be along OY. Then, the equation of the parabola is

$x^2 = 4ay$ …(i)



The coordinates end A of the arc are (2.5, 10) and it lies on the parabola (i)

$\therefore \quad (2.5)^2 = 4a \times 10$

$$a = \frac{6.25}{40} = \frac{625}{4000} = \frac{5}{32}$$

Putting the value of a in (i), we obtain that the equation of the parabolic arc is

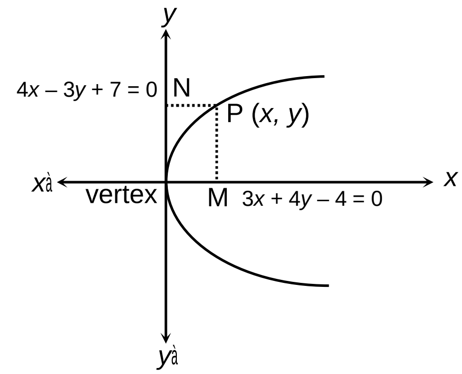
$x^2 = \frac{5}{8}y$, when y = 2, we have $x^2 = \frac{5}{8} \times 2$ $\quad x = \frac{\sqrt{5}}{2}$ m

Hence, the width of the arc at a height of 2m from the vertex is $2 \times \frac{\sqrt{5}}{2}$ m = $\sqrt{5}$ m

Illustration 11

Question: **Find the equation of the parabola whose latusrectum is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.**

Solution: Let $P(x, y)$ be any point on the parabola and let PM and PN be perpendicular from P on the axis and tangent at the vertex respectively. Then



$$PM^2 = (\text{Latusrectum})\,(PN)$$

$$\left(\frac{3x + 4y - 4}{\sqrt{3^2 + 4^2}}\right)^2 = 4\left(\frac{4x - 3y + 7}{\sqrt{4^2 + (-3)^2}}\right)$$

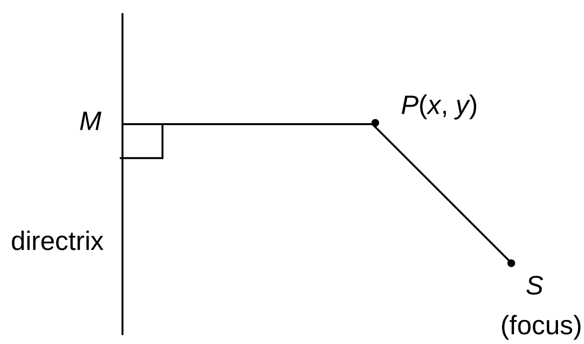
$$(3x + 4y - 4)^2 = 20(4x - 3y + 7)$$

This is the equation of the required parabola.

3. ELLIPSE

3.1 DEFINITION

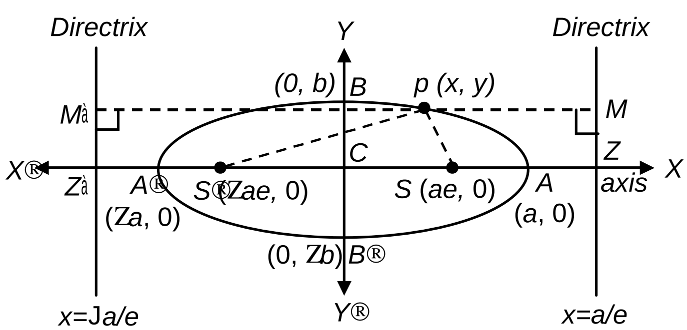
If the ratio of the distance of a point from a fixed point on the plane (focus) to the distance of the point from a fixed line in the plane (directrix) bears a constant ratio less than one, the locus of the point describes an ellipse. The constant ratio is called the eccentricity of the ellipse.



i.e., $\frac{PS}{PM} = e < 1.$

3.2 EQUATION OF AN ELLIPSE IN STANDARD FORM

Let S be the focus and ZM the directrix of the ellipse. Draw $SZ \perp ZM$. A and A′ divide SZ internally and externally in the ratio e : 1 (e < 1).



Therefore, $SA = eAZ$ …(i)

and $\quad SA' = eA'Z \quad \ldots(ii)$

A and A' lie on the ellipse. Let $AA' = 2a$ and C the mid point of AA' be the origin

$$CA = CA' = a \quad \ldots(iii)$$

Let P (x, y) be any point on the ellipse referred to CA and CB as coordinate axes. Then adding (i) and (ii).

$$SA + SA' = e\,(AZ + A'Z)$$

$$AA' = e\,(CZ - CA + CA' + CZ)$$

$$AA' = e\,(2CZ) \quad (\because CA = CA')$$

$$2a = 2eCZ$$

$\therefore \quad CZ = a/e$

$\therefore$ The directrix MZ is $\quad x = CZ = a/e$

or $\quad \dfrac{a}{e} - x = 0 \qquad \because e < 1, \therefore \dfrac{a}{e} > 1$

and subtracting (i) and (ii),

$$SA' - SA = e\,(A'Z - AZ)$$

$$(CA' + CS) - (CA - CS) = e\,(AA')$$

$$2CS = e\,(AA')$$

$$2CS = e\,(2a) \quad (\because CA = CA')$$

$\therefore \quad CS = ae$

$\therefore$ The focus S is $(CS, 0)$ i.e., $(ae, 0)$

Now draw $PM \perp MZ$

$\therefore \quad \dfrac{SP}{PM} = e$

or $\quad (SP)^2 = e^2\,(PM)^2$

$$(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$(x - ae)^2 + y^2 = (a - ex)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2\,(1 - e^2) + y^2 = a^2\,(1 - e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

or $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2\,(1 - e^2)$

This is the standard equation of an ellipse. AA' and BB' are called the major and minor axes of the ellipse. (Here $b < a$) and A and A' are the vertices of the ellipse.

3.3 IMPORTANT POINTS

On the negative side of origin take a point S' which is such that

$$CS = CS' = ae$$

and another point Z' then

$$CZ = CZ' = \frac{a}{e}$$

$\therefore$ Coordinates of S' are $(-ae, 0)$ and equation of second directrix (i.e., $Z'M'$) is

$$x = -\frac{a}{e}$$

Let $P(x, y)$ be any point on the ellipse then

$$S'P = e\,PM'$$

or $\quad (S'P)^2 = e^2\,(PM')^2$

or $\quad (x + ae)^2 + (y - 0)^2 = e^2\left(x + \frac{a}{e}\right)^2$

or $\quad (x + ae)^2 + y^2 = (ex + a)^2$

or $\quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

or $\quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

or $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad$ where $b^2 = a^2(1 - e^2)$

The equation being the same as that of the ellipse when S $(ae, 0)$ is focus and MZ i.e., $x = a/e$ is directrix. So, every ellipse has two foci and two directrices.

Hence coordinates of foci are $(\pm ae, 0)$ and equations of directrices are $x = \pm a/e$.

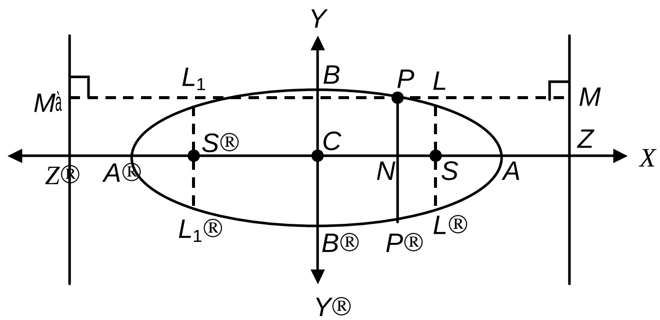
Distance between foci $SS' = 2ae$ and distance between directrices $ZZ' = 2a/e$.

3.4 TERMS RELATED TO ELLIPSE

Let the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$

Centre: All chords passing through C get bisected at C.

Here $C \equiv (0, 0)$



Foci: S and S' are two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$.

Directrices: ZM and $Z'M'$ are two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

Axes: The lines AA' and BB' are called the major and minor axes of the ellipse.

Double ordinates: If P be a point on the ellipse, draw PN perpendicular to the axis of the ellipse and produced to meet the curve again at P'. Then PP' is called a double ordinate.

If abscissa of P is h, then ordinate of P is

$$\frac{y^2}{b^2} = 1 - \frac{h^2}{a^2}$$

… $\quad y = \frac{b}{a}\sqrt{(a^2 - h^2)}$ (for I[st] quadrant)

and ordinate of P' is $\quad y = -\frac{b}{a}\sqrt{(a^2 - h^2)}$ (for IV[th] quadrant)

Hence coordinates of P and P' are

$$h, \frac{b}{a}\sqrt{a^2 - h^2} \quad \text{and} \quad h, -\frac{b}{a}\sqrt{a^2 - h^2} \quad \text{respectively.}$$

Latus rectum: The double ordinates LL' and L_1L_1' are the latus rectum of the ellipse. These lines are perpendicular to major axis $A'A$ and through the foci S and S' respectively.

Length of the latus rectum

Now let $\qquad LL' = 2k$

then $\qquad LS = L'S = k$

Coordinates of L and L' are (ae, k) and $(ae, -k)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{a^2e^2}{a^2} + \frac{k^2}{b^2} = 1 \qquad k^2 = b^2(1 - e^2) = b^2 \; \frac{b^2}{a^2} \qquad [\because b^2 = a^2(1 - e^2)]$$

$$k = \frac{b^2}{a} \qquad (\because k > 0)$$

$$2k = \frac{2b^2}{a} = LL'$$

$\therefore$ Length of latus rectum $LL' = L_1L_1' = \dfrac{2b^2}{a}$ and end points of latus-rectum are

$$L \equiv \left(ae, \frac{b^2}{a}\right);\ L' \equiv \left(ae, -\frac{b^2}{a}\right) \text{ and } L_1 \equiv \left(-ae, \frac{b^2}{a}\right);\ L_1' \equiv \left(-ae, -\frac{b^2}{a}\right)$$

respectively.

Focal chord: A chord of the ellipse passing through its focus is called a focal chord.

Vertices: The vertices of the ellipse are the points where the ellipse meets its major axis. Here A and A' are the vertices.

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

Parametric equation of the ellipse

Let the equation of ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then $x = a\cos\theta$, $y = b\sin\theta$ are called parametric equations of the ellipse.

3.5 OTHER FORM OF THE ELLIPSE

In this case major and minor axis of the ellipse along y-axis and x-axis respectively.

then $AA' = 2b$ and $BB' = 2a$ $\quad (b > a)$

The foci S and S' are $(0, be)$ and $(0, -be)$ respectively.

The directrices are MZ and $M'Z'$ are $y = \dfrac{b}{e}$ and $y = -\dfrac{b}{e}$ respectively.

The equation of ellipse again is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

But now, $a^2 = b^2(1 - e^2)$

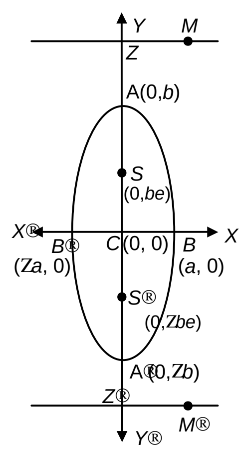


Illustration 12

Question: **Find the length of major axis, minor axis of the ellipse $x^2 + 16y^2 = 16$.**

Solution: $x^2 + 16y^2 = 16 \qquad \dfrac{x^2}{16} + \dfrac{y^2}{1} = 1 \qquad \dfrac{x^2}{4^2} + \dfrac{y^2}{1^2} = 1$

length of major axis $= 2a = 8$ units.

length of minor axis $= 2b = 2$ units.

Illustration 13

Question: **Find the equation of the ellipse whose focus is the point (6, 7), directrix $x + y + 2 = 0$ and eccentricity $\frac{1}{\sqrt{3}}$.**

Solution: By definition of an ellipse, we have $SP = e\,PM$.

$$\therefore \quad \sqrt{[(x-6)^2 + (y-7)^2} = \frac{1}{\sqrt{3}} \cdot \left| \frac{x+y+2}{\sqrt{(1+1)}} \right|$$

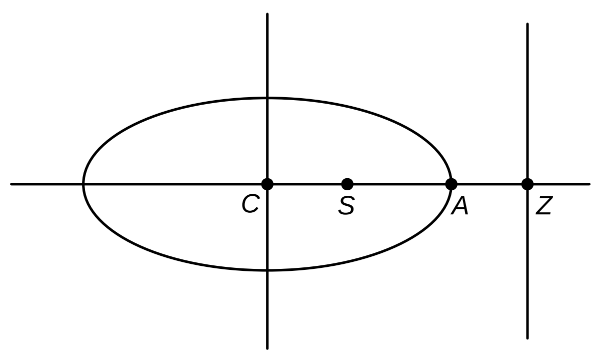
Squaring, $6(x^2 + y^2 - 12x - 14y + 85) = x^2 + y^2 + 4 + 2xy + 4y + 4x$

or $\quad 5x^2 + 5y^2 - 76x - 88y - 2xy + 506 = 0$

Illustration 14

Question: **Find the equation of an ellipse having its centre at the point (2, -3), one focus at (3, -3) and one vertex at (4, -3).**

Solution: Given an ellipse having $C(2, -3)$ as centre, $S(3, -3)$ as focus and $A(4, -3)$ as a vertex



CA = the semi-major axis

$$= \sqrt{[(4-2)^2 + (-3+3)^2]} = 2$$

$\therefore \quad a = 2$.

Also $CS = ae$, which gives $e = \frac{1}{2}$

Thus A divides SZ in the ratio $\frac{1}{2} : 1$ i.e., $1 : 2$.

Let Z be (h, k). Then $h = 6$, $k = -3$

Since CA is parallel to the x-axis so directrix is parallel to y-axis through $(6, -3)$, i.e. directrix is $x = 6$.

Hence by the definition of an ellipse, the required equation is

$$(x-3)^2 + (y+3)^2 = \frac{1}{4}(x-6)^2 \quad \text{or} \quad 3x^2 + 4y^2 - 12x + 24y + 36 = 0$$

Illustration 15

Question: **Find the eccentricity foci and the length of the latus rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$.**

Solution: The given equation of the ellipse is $x^2 + 4y^2 + 8y - 2x + 1 = 0$

$$x^2 - 2x + 4y^2 + 8y = -1 \qquad (x^2 - 2x + 1) + 4(y^2 + 2y + 1) = -1 + 1 + 4$$

$$(x-1)^2 + 4(y+1)^2 = 4 \qquad \frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{1^2} = 1 \qquad \ldots(i)$$

Shifting the origin to (1, –1) without rotating the axes and denoting the new coordinates with respect to these axes by *X* and *Y*, we have

$$x = X + 1,\ y = Y - 1 \qquad \ldots(ii)$$

Using these relations equation (i) reduces to $\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

On comparing, we get $a^2 = 2^2$, $b^2 = 1$ $\quad a = 2, b = 1$

Let e be the eccentricity of the ellipse. Then $b^2 = a^2(1 - e^2) \quad e = \frac{\sqrt{3}}{2}$

The coordinates of foci w.r.t. new axes are $(X = \pm ae,\ y = 0)$ i.e., $(X = \pm\sqrt{3},\ Y = 0)$

So, coordinates of foci w.r.t. old axes are $(1 \pm \sqrt{3},\ -1)$

Length of the latus rectum = $2\frac{b^2}{a} = \frac{2(1)}{2} = 1$

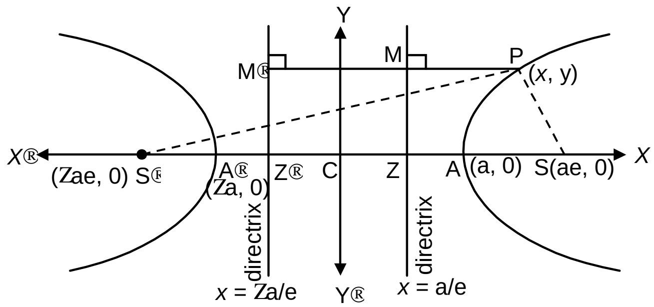
4. HYPERBOLA

4.1 DEFINITION

A hyperbola is a set of all points in a plane whose distances from focus and their distances from the directrix bear a constant ratio, greater than one.

4.2 EQUATION OF A HYPERBOLA IN STANDARD FORM

Let S be the focus and ZM the directrix of the hyperbola. Draw $SZ \perp ZM$. Let A and A' divide SZ internally and externally in the ratio $e : 1$ ($e > 1$).



Then $\quad SA = e\,AZ \qquad \ldots$(i)

and $\quad SA' = e\,A'Z \qquad \ldots$(ii)

Clearly A and A' will lie on the hyperbola. Let $AA' = 2a$ and take C the mid point of AA' as origin.

$\therefore \quad CA = CA' = a$

Let P (x, y) be any point on the hyperbola and CA as x-axis, the line through C perpendicular to CA as y-axis.

Then adding (i) and (ii) $\quad SA + SA' = e\,(AZ + A'Z)$

$$CS - CA + CS + CA' = e\,(AA')$$

$$2CS = e\,(2a) \qquad (\because CA = CA')$$

$\therefore \quad CS = ae$

$\therefore$ The focus S is (CS, 0), i.e., (ae, 0) and subtracting (i) from (ii),

$$SA' - SA = e\,(A'Z - AZ)$$

$$AA' = e\,[(CA' + CZ) - (CA - CZ)]$$

$$AA' = e\,(2CZ) \qquad (\therefore CA = CA')$$

$$2a = e\,(2CZ)$$

$\therefore \quad CZ = a/e$

$\therefore$ The directrix MZ is $x = CZ = a/e$

or $\quad x - a/e = 0 \qquad \because e > 1, \therefore \frac{a}{e} < 1$

Now draw $PM \perp MZ$.

$\therefore \qquad \frac{SP}{PM} = e$ or $(SP)^2 = e^2 (PM)^2$

or $\qquad (x - ae)^2 + (y - 0)^2 = e^2 \left(x - \frac{a}{e}\right)^2 \qquad (x - ae)^2 + y^2 = (ex - a)^2$

$$x^2 + a^2 e^2 - 2aex + y^2 = e^2 x^2 - 2aex + a^2$$

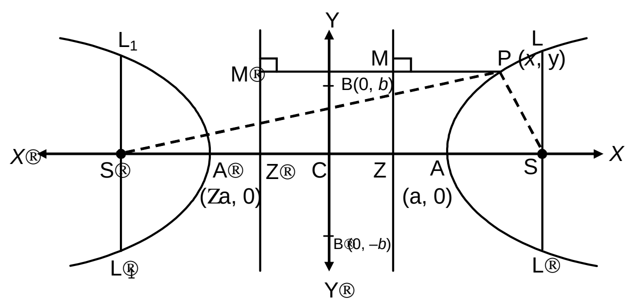
$$x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1) \qquad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

or $\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2 (e^2 - 1)$

This is the standard equation of the hyperbola.

4.3 IMPORTANT TERMS

Centre: All chords passing through C (0, 0) get bisected at C.



Foci and directrices: S and S' are the foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively and ZM and $Z'M'$ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

Axes: The points A $(a, 0)$ and A' $(-a, 0)$ are called the vertices of the hyperbola and line AA' is called the transverse axis and the line perpendicular to it through the centre (0, 0) of the hyperbola is called conjugate axis.

The length of transverse and conjugate axes is taken as $2a$ and $2b$ respectively.

Latus rectum: Double ordinates LL' and L_1L_1' are the latus rectum of the hyperbola. These lines are perpendicular to transverse axis AA' and through the foci S and S' respectively.

Length of latus rectum

Now let $\qquad LL' = 2k$

then $\qquad LS = L'S = k$

coordinates of L and L' are (ae, k) and $(ae, -k)$ respectively, lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \qquad \frac{a^2e^2}{a^2} - \frac{k^2}{b^2} = 1$ or $k^2 = b^2 (e^2 - 1)$

$$= b^2 \; \frac{b^2}{a^2} \qquad [\because b^2 = a^2 (e^2 - 1)]$$

$$\therefore \qquad k = \frac{b^2}{a} \qquad (\because k < 0)$$

$$\therefore \qquad 2k = \frac{2b^2}{a} = LL'$$

$\therefore$ Length of latus rectum $LL' = L_1L_1' = \frac{2b^2}{a}$ and

end points of latus rectum are

$$L \equiv \left(ae, \frac{b^2}{a}\right);\ L' \equiv \left(ae, \frac{-b^2}{a}\right);\ L_1 \equiv \left(-ae, \frac{b^2}{a}\right);\ L_1' \equiv \left(-ae, -\frac{b^2}{a}\right)$$

respectively.

4.4 PARAMETRIC EQUATIONS OF THE HYPERBOLA

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. then $x = a\sec\theta$ and $y = b\tan\theta$ are known as the parametric equations.

Illustration 16

Question: **Find the equation of the hyperbola in standard form whose eccentricity is $\sqrt{2}$ and the distance between whose foci is 16.**

Solution: If S and S′ be the foci, then SS′ = $2ae = 2a\sqrt{2} = 16$

as $e = \sqrt{2}$, so a = $4\sqrt{2}$

Also, $b^2 = a^2(e^2 - 1)$ gives $b^2 = 32(2 - 1) = 32$

Hence the equation of the hyperbola is $\frac{x^2}{32} - \frac{y^2}{32} = 1$ i.e., $x^2 - y^2 = 32$

Illustration 17

Question: **Find the equation of the standard hyperbola whose conjugate axis is 5 and the distance between the foci is 13.**

Solution: If a and b are the semi axes, then we have $2ae = 13$, $2b = 5$.

Now $a^2 = a^2e^2 - b^2 = \frac{1}{4}(169 - 25) = 36$ $\qquad a = 6$

Hence the required equation of the hyperbola is $\frac{x^2}{36} - \frac{4y^2}{25} = 1$

or $25x^2 - 144y^2 = 900$.

PRACTICE PROBLEMS

PP1. Find the centre and radius of the following circles:

(i) $1 - x^2 - y^2 = 0$ (ii) $x^2 + y^2 - 2x + 4y = 8$

PP2. Find the equation of circle whose centre is (2, –3) and which passes through the intersection of lines $3x + 2y = 11$ and $2x + 3y = 4$.

PP3. Find the equation of circle that passes through the points (1, 0), (–1, 0) and (0, 1).

PP4. Find the equation of the circle concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 18 = 0$ and passing through the point (–4, –5).

PP5. Find the equation of circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively.

PP6. Find the co-ordinates of focus, axis, the equation of directrix and latus rectum of parabola

(i) $y^2 = 8x$ (ii) $x^2 = -16y$ (iii) $y^2 + 4x + 4y - 3 = 0$ (iv) $x^2 + y = 6x - 14$

PP7. Find the equation of parabola passing through (5, 2) with vertex (0, 0) and symmetric with respect to *y*-axis.

PP8. Find the co-ordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the following ellipse:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (ii) $36x^2 + 4y^2 = 144$

(iii) $x^2 + 2y^2 - 2x + 12y + 10 = 0$ (iv) $x^2 + 4y^2 - 2x = 0$

PP9. Find the equation of the ellipse whose focus is (1, –2), corresponding directrix is $3x - 2y + 5 = 0$ and eccentricity equal to $1/2$.

PP10. A rod *AB* (=*l* cm) rests in between two co-ordinate axes in such a way that the end point *A* lies on *x*-axis and end point *B* lies on *y*-axis. *A* point *P*(*x, y*) is taken on the rod in such a way that *AP : PB* = 2 : 3. Show that locus of *P* is an ellipse.

PP11. Find the co-ordinates of the foci, the vertices, the eccentricity and the length of latus rectum of the hyperbolas:

(i) $16x^2 - 9y^2 = 576$ (ii) $5y^2 - 9x^2 = 36$ (iii) $x^2 - 2y^2 - 2x + 8y - 1 = 0$

PP12. Find the equation of the hyperbola satisfying the given condition:

Vertices (±2, 0), foci (±3, 0)

PP13. Find the equation of the hyperbola satisfying the given condition:

Foci (0, ±13), the conjugate axis is of length 24.

PP14. Find the equation of the hyperbola satisfying the given condition:

Foci $(\pm 3\sqrt{5}, 0)$ the latus-rectum is of length 8.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi-minor axis is $\sqrt{5}$.

Solution:

Let S and S' be two foci of the required ellipse.

Then the coordinates of S and S' are (2, 3) and (–2, 3) respectively.

$\therefore \quad SS' = 4$

Let 2a and 2b be the lengths of the axes of the ellipse and e be its eccentricity

Then $SS' = 2ae \quad \Rightarrow \quad 2ae = 4 \quad \Rightarrow \quad ae = 2$

Now, $b^2 = a^2(1-e^2) \;\Rightarrow\; 5 = a^2 - 2^2 \;\Rightarrow\; a = 3$

Let $P(x, y)$ be any point on the ellipse. Then

$$SP + S'P = 2a$$

$$\Rightarrow \quad \sqrt{(x-2)^2+(y-3)^2} + \sqrt{(x+2)^2+(y-3)^2} = 6$$

$$\Rightarrow \quad [(x-2)^2+(y-3)^2] - [(x+2)^2+(y-3)^2] = 36 - 12\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$$

$$\Rightarrow \quad -8x = 36 - 12\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$$

$$\Rightarrow \quad (2x-9)^2 = 9\{(x+2)^2+(y-3)^2\} \;\Rightarrow\; 5x^2+9y^2+72x-54y+36=0$$

This is the required equation of the ellipse.

***Example* 2:**

Find the equation of the circle, which touches both the axes and the straight line 4*x* + 3*y* = 6 in the first quadrant and lies below it.

***Solution*:**

If a is the radius of the circle which touches both the axes and lies in the first quadrant, then its centre will be $C(a, a)$.

If the circle also touches the line $4x + 3y = 6$, then length of $\perp$ from centre (a, a) on this line is equal to radius of circle,

i.e., $\dfrac{4a+3a-6}{\sqrt{(16+9)}} = \pm\, a \Rightarrow a = 1/2, 3.$

$\therefore$ For the circle which lies below the line $a = 1/2$

Hence, the equation of the required circle is $(x \text{ J } a)^2 + (y \text{ J } a)^2 = a^2$

or $x^2 + y^2 - 2ax$ J $2ay + a^2 = 0$ or $x^2 + y^2 - x$ J $y + 1/4 = 0$ or $4x^2 + y^4 - 4x$ J $4y + 1 = 0$

***Example* 3:**

Find the centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$.

***Solution*:**

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

It passes through the points (0, 0) and (1, 0),

$\therefore$ $c = 0$ and $1 + 2g + c = 0$

$\therefore$ $g = -1/2$.

Radius of circle (i) is

$$r_1 = \sqrt{(g^2 + f^2 - c)} = \sqrt{(1/4 + f^2)}.$$

The centre of the circle $x^2 + y^2 = 9$ ...(ii)

is (0, 0) and radius $r_2 = 3$.

Since the circle (i) passes through the centre (0, 0) of circle (ii) and it also touches the circle (ii) internally.

$\therefore$ Diameter of circle (i) = radius of circle (ii), *i.e,* $2r_1 = r_2$

or $2\sqrt{(1/4 + f^2)} = 3 \Rightarrow f^2 = 2 \Rightarrow f = \pm\sqrt{2}$.

Hence the centre of circle (i) is ($-g$, Jf), *i.e.,* $(1/2, -\sqrt{2})$ or $(1/2, \sqrt{2})$.

***Example* 4:**

Find the equations of the circles passing through (–4, 2) and touching the lines $x + y = 2$ and $x - y = 2$.

***Solution*:**

A circle touching the given lines must have its centre on one or the other of the bisectors, *i.e*., on $y = 0$ or $x = 2$. When the circle passes through (–4, 2), the centre cannot lie on the line $x = 2$. Hence the centre lies on the x-axis.

Using (h, 0) for the coordinates of the centre, we have $\frac{(h-2)^2}{2}$ = the square of the length of the radius, since the perpendicular from (h, 0) to the line $x + y = 2$ is equal in length to the radius of the circle.

Also, $(h + 4)^2 + 4$ = the square on the length of the radius, since the circle passes through (–4, 2).

$\therefore$ we have $(h + 4)^2 + 4 = \frac{(h-2)^2}{2}$

Rearranging, $h^2 + 20h + 36 = 0$

$\therefore$ $h = -2$ or -18

$\therefore$ the equation of one circle is $(x + 2)^2 + y^2 = 8$ or $x^2 + y^2 + 4x - 4 = 0$ and the equation of another circle is $(x + 18)^2 + y^2 = 200$ or $x^2 + y^2 + 36x + 124 = 0$.

***Example* 5:**

Find the equation of the parabola whose focus is (1, 1) and the directrix is *x* + *y* + 1 = 0.

Solution:

Let $P(x, y)$ be any point on the parabola.

Then the distance of (x, y) from the focus (1, 1).

= distance of $P(x, y)$ from the directrix $(x + y + 1 = 0)$

$$\therefore \quad \sqrt{(x-1)^2+(y-1)^2} = \frac{|x+y+1|}{\sqrt{(1)^2+(1)^2}} \qquad \dots(i)$$

Squaring (i), we get $(x-1)^2+(y-1)^2 = \left(\frac{x+y+1}{\sqrt{2}}\right)^2$

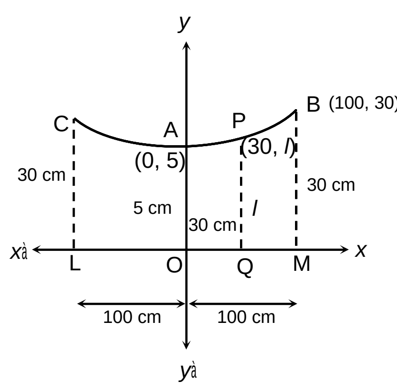
or $\quad 2[x^2+1-2x+y^2+1-2y] = x^2+y^2+2xy+2y+2x+1$ or $x^2-2xy+y^2-6x-6y+3=0$

This is the required of the parabola.

***Example* 6:**

The towers of a bridge, hung in the form of a parabola, have their tops 30 m above the roadway and are 200 metres apart. If the cable is 5 m above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.

***Solution*:**



Let *CAB* be the bridge and $X'OX$ be the roadway. Let *A* be the centre of the bridge.

Taking $X'OX$ as *x*-axis and *y*-axis along *OA*, we find that the coordinates of *A* are (0, 5).

Clearly, the bridge is in the shape of a parabola having its vertex at *A*(0, 5). Let its equation be

$$x^2 = 4a(y-5) \qquad \dots(i)$$

It passes through *B*(100, 30). Therefore,

$$(100)^2 = 4a(30-5) \Rightarrow a = 100$$

Putting the value of *a* in (i), we get

$$x^2 = 400(y-5) \qquad \dots(ii)$$

Let *l* metres be the length of the vertical supporting cable 30 m from the centre.

Then, *P*(30, *l*) lies on (ii). Therefore, $900 = 400(l-5) \Rightarrow l = \frac{9}{4}+5 = \frac{29}{4}$ m

Hence, the length of the vertical supporting cable 30 m from the centre of the bridge is $\frac{29}{4}$ m.

Example 7:

Obtain the equation of a hyperbola with coordinate axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units.

Solution:

Let equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$ …(i)

If vertices are $A(a, 0)$ and A' $(-a, 0)$ and foci are S $(ae, 0)$ and S' $(-ae, 0)$

given $l\,(S'A) = 9$ and $l\,(SA) = 1$

$\Rightarrow$ $a + ae = 9$ and $ae - a = 1$ or $a(1 + e) = 9$ and or $a\,(e - 1) = 1$

$\therefore$ $\frac{a(1+e)}{a(e-1)}=\frac{9}{1}$ $\Rightarrow$ $1 + e = 9e - 9$ $\Rightarrow$ $e=\frac{5}{4}$

$\because$ $a\,(1 + e) = 9$; $\therefore$ $a\left(1+\frac{5}{4}\right)=9$ $\Rightarrow$ $a = 4$

$b^2 = a^2\,(e^2 - 1) = 16\left(\frac{25}{16}-1\right)$; $\therefore$ $b^2 = 9$

from (i), equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$.

Example 8:

Prove that the equation $\sqrt{(x+4)^2+(y+2)^2}-\sqrt{(x-4)^2+(y-2)^2}=8$ represents a hyperbola.

Solution:

$[(x + 4)^2 + (y + 2)^2] - [(x - 4)^2 + (y - 2)^2] = 16x + 8y$ …(i)

$\sqrt{(x+4)^2+(y+2)^2}-\sqrt{(x-4)^2+(y-2)^2}=8$ …(ii)

Dividing (i) by (ii), we get $\sqrt{(x+4)^2+(y+2)^2}-\sqrt{(x-4)^2+(y-2)^2}=\frac{16x+8y}{8}$ …(iii)

From (ii) and (iii), $2\sqrt{(x+4)^2+(y+2)^2}=\frac{16x+8y+64y}{8}$

$$2\sqrt{(x+4)^2+(y+2)^2}=\frac{8x+4y+32y}{8}=\frac{\sqrt{5}}{2}\left(\frac{2x+y+8}{\sqrt{5}}\right)$$

Clearly it is a hyperbola with eccentricity $\frac{\sqrt{5}}{2}$.

***Example* 9:**

Find the equation of the hyperbola, the distance between whose foci is 16, whose eccentricity is $\sqrt{2}$ and whose axis is along the *x*-axis with the origin as its centre.

Solution:

We have $b^2 = a^2 (e^2 - 1) = a^2 \Rightarrow b = a$

Also $2ae = 16 \Rightarrow ae = 8 \Rightarrow a = 4\sqrt{2}$

Hence the equation of the required hyperbola is $\frac{x^2}{32} - \frac{y^2}{32} = 1 \Rightarrow x^2 - y^2 = 32$

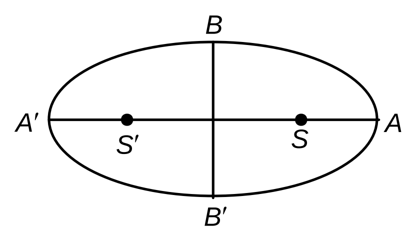
***Example* 10:**

Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

Solution:

Let the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (let $a > b$)

Then the foci are S (ae, 0) and S′ ($-ae$, 0), length of minor axis $BB' = 2b$ and length of latus rectum = $\frac{2b^2}{a}$.



$\therefore$ According to the question

$BB' = SS'$

$\Rightarrow 2b = 2ae \Rightarrow b = ae$ …(i)

and $\frac{2b^2}{a} = 10$

$\Rightarrow b^2 = 5a$ …(ii)

also we have $b^2 = a^2 (1 - e^2)$ …(iii)

Putting the value of b from equation (i) in (iii), we have $a^2e^2 = a^2 (1 - e^2)$

$\Rightarrow e^2 = 1 - e^2 \Rightarrow 2e^2 = 1$

$\therefore e = \frac{1}{\sqrt{2}}$

from equation of (i) we have $b = \frac{a}{\sqrt{2}}$

$\therefore b^2 = \frac{a^2}{2} \Rightarrow 5a = \frac{a^2}{2}$ { from equation (ii)}

$\Rightarrow a = 10$

From equation (ii), $b^2 = 5 \times 10 = 50$

Putting the values of a and b in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of required ellipse is

$\frac{x^2}{100}+\frac{y^2}{50}=1$ or $x^2 + 2y^2 = 100$

EXERCISE – I

1. (i) Find the equation of the circle with centre (0, –1) and radius 1.

(ii) Find the equation of the circle with centre (–a, –b) and radius $\sqrt{a^2 - b^2}$.

(iii) Find the equation of the circle whose centre is (h, k) and which passes through the point (p, q).

(iv) Prove that the centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and

$x^2 + y^2 + 2x + 4y - 10 = 0$ and $x^2 + y^2 - 10x - 16y - 1 = 0$ are collinear.

2. (i) Find the equation of the circle passing through the points (20, 3), (19, 8) and (2, –9).

(ii) Find the equation of the circle passing through the vertices of the triangle whose sides are along $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$.

(iii) Show that the four points (0, 0), (1, 1), (5, –5) and (6, –4) are concyclic.

(iv) Find the equation of a circle passing through (0, 0), (a, 0) & (0, b).

(v) Find the equation of the circle which circumscribes the triangle formed by lines $x = 0$, $y = 0$ and $lx + my = 1$.

3. (i) Find the equation of the circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ and whose centre is the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.

(ii) Find the equation of the circle which has its centre at the point (3, 4) and touches the straight line $5x + 12y - 1 = 0$.

(iii) Find the equation of the circle which passes through the points (–1, 2) and (3, –2) and whose centre lies on the line $x - 2y = 0$.

4. Find the equations of the circles which pass through the origin and cut off equal chords of $\sqrt{2}$ units from the lines $y = x$ and $y = -x$.

5. (i) Find the length of tangent drawn from the point (6, –7) to the circle

$3x^2 + 3y^2 - 7x - 6y = 12$.

(ii) If the lengths of tangents from the point (1, 2) to the circles $x^2 + y^2 + x + y - 4 = 0$ & $3x^2 + 3y^2 - x - y - k = 0$ be in the ratio 3 : 4, find the value of k.

6. Find the co-ordinates of the focus and the equation of the directrix in each of the following parabolas :

(i) $x^2 = 6y$ (ii) $4y^2 + 12x - 12y + 39 = 0$

7. (i) Find the equation of the parabola whose focus is (0, 0) and the directrix $2x - y - 1 = 0$.

(ii) Find the equation of parabola if the focus is at (0, 0) and vertex is at the intersection of the lines $x + y = 1$ and $x - y = 3$.

8. Find the area of triangle formed by lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

9. For the parabola $y^2 = 4ax$, find the extremities of a double ordinate of length $8a$ and prove that the lines from the vertex to its extremities are at right angle.

10. Find the lengths of the major and the minor axes, the coordinates of the foci, the vertices the eccentricity and the equations of the directrices of the ellipse $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

EXERCISE – II

1. Find the lengths of the major and the minor axes, the coordinates of the foci, the vertices the eccentricity and the equations of the directrices of the ellipse $4x^2 + 9y^2 = 1$.

2. Find the equation of the ellipse in the following case:

foci at $(\pm 2, 0)$ and $e = \frac{1}{2}$.

3. Find the equation of the ellipse in the following case :

eccentricity $e = \frac{2}{3}$ and length of the latus rectum = 5.

4. Find the equation of the ellipse in the standard form whose minor axis is equal to the distance between foci and whose latus rectum is 10.

5. Find the equation of the ellipse whose vertices are $(0, \pm 10)$ and eccentricity $e = \frac{4}{5}$.

6. Find the coordinates of the vertices, the foci, the eccentricity and the equations of the directrices of the hyperbola $3x^2 - 2y^2 = 1$.

7. Find the equation of the set of all points such that the difference of their distances from (4, 0) and (–4, 0) is always equal to 2.

8. Find the equation of the hyperbola whose vertices are $(0, \pm 5)$ and foci at $(0, \pm 8)$.

9. Find the equation of the hyperbola :

whose focus is (2, 1), directrix is $2x + 3y = 1$ & $e = 2$.

10. Find the axes, eccentricity, latus rectum and the coordinates of the foci of the hyperbola $25x^2 - 36y^2 = 225$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. (i) (0, 0), 1 (ii) (1, –2), $\sqrt{13}$

PP2. $x^2 + y^2 - 4x + 6y + 3 = 0$

PP3. $x^2 + y^2 = 1$

PP4. $x^2 + y^2 - 4x - 6y - 87 = 0$

PP5. $\left(x - \frac{3}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2$

PP6.

	(i)	(ii)	(iii)	(iv)
Focus	(2, 0)	(0, –4)	$(3/4, -2)$	$(3, -21/4)$
Directrix:	$x = -2$	$y = 4$	$4x = 11$	$4y + 19 = 0$
Latus rectum	8	16	4	1
Axis	x-axis	y-axis	y + 2 = 0	$x = 3$

PP7. $2x^2 = 25y$

PP8.

	(i)	(ii)	(iii)	(iv)
Foci	(±4, 0)	$(0, \pm 4\sqrt{2})$	$\left(1 \pm \frac{\sqrt{3}}{2}, -3\right)$	$\left(1 \pm \frac{\sqrt{3}}{2}, 0\right)$
Vertices	(±5, 0)	(0, ±6)	(4, –3) & (–2, –3)	(2, 0) & (0, 0)
Major axis	10 units	12 units	6	2
Minor axis	6 units	4 units	$3\sqrt{2}$	1
Eccentricity	$\frac{4}{5}$	$\frac{2\sqrt{2}}{3}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
Latus rectum	$\frac{18}{5}$	$\frac{4}{3}$	3	$\frac{1}{2}$

PP9. $43x^2 + 48y^2 + 12xy - 134x + 228y + 235 = 0$

PP11.

	(i)	(ii)	(iii)
Foci	(±10, 0)	$\left(0, \pm\dfrac{2\sqrt{14}}{\sqrt{5}}\right)$	(1, 2±3)
Vertices	(±6, 0)	$\left(0, \pm 6\dfrac{6}{\sqrt{5}}\right)$	$\left(1,\ 2\pm\sqrt{3}\right)$
Eccentricity	$\dfrac{5}{3}$	$\dfrac{\sqrt{14}}{3}$	$\sqrt{3}$
Latus rectum	$\dfrac{64}{3}$	$\dfrac{4\sqrt{5}}{3}$	$4\sqrt{3}$

PP12. $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

PP13. $\dfrac{y^2}{25} - \dfrac{x^2}{144} = 1$

PP14. $\dfrac{x^2}{25} - \dfrac{y^2}{20} = 1$

ANSWERS TO EXERCISE – I

1. (i) $x^2 + y^2 + 2y = 0$

(ii) Centre (2, –3) and radius 7

(iii) $x^2 + y^2 - 2hx \text{ J } 2ky \text{ J } p^2 \text{ J } q^2 + 2hp + 2kq = 0$

2. (i) $x^2 + y^2 - 14x \text{ J } 6y - 111 = 0$

(ii) $x^2 + y^2 + 4x + 6y \text{ J } 12 = 0$

(iv) $x^2 + y^2 - ax \text{ J } by = 0$

(v) $x^2 + y^2 - \frac{1}{l}x - \frac{1}{m}y = 0$

3. (i) $x^2 + y^2 + 4x \text{ J } 2y = 0$

(ii) $169(x^2 + y^2 - 6x \text{ J } 8y) + 381 = 0$

(iii) $x^2 + y^2 - 4x \text{ J } 2y \text{ J } 5 = 0$

4 $x^2 + y^2 \pm 2y = 0$, $x^2 + y^2 \pm 2x = 0$

5. (i) 9 units

(ii) J4

6. (i) $\left(0, \frac{3}{2}\right), y = \frac{-3}{2}$ (ii) $\left(-\frac{13}{4}, \frac{3}{2}\right), x = -\frac{7}{4}$

7. (i) $x^2 + 4y^2 + 4xy + 4x - 2y - 1 = 0$

(ii) $(x + 2y)^2 + 40x - 20y - 100 = 0$

8. 18 sq. units

9. $(4a, 4a)$ and $(4a, -4a)$

10. 26, 24, (±5 , 0), (±13, 0), $\frac{15}{3}$, $x = \pm \frac{169}{5}$

ANSWERS TO EXERCISE – II

1. $1, \frac{2}{3}, \left(\pm\frac{\sqrt{5}}{6}, 0\right), \left(\pm\frac{1}{2}, 0\right), \frac{\sqrt{5}}{3}$

2. $3x^2 + 4y^2 = 48$

3. $20x^2 + 36y^2 = 405$

4. $x^2 + 2y^2 = 100$

5. $100x^2 + 36y^2 = 3600$

6. $\left(\pm\frac{1}{\sqrt{3}}, 0\right), \left(\pm\sqrt{\frac{5}{6}}, 0\right)$, $\frac{\sqrt{5}}{\sqrt{2}}, x = \pm\sqrt{\frac{2}{15}}$

7. $15x^2 - y^2 = 15$

8. $\frac{y^2}{25} - \frac{x^2}{39} = 1$

9. $3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$

10. Transverse axis 6, Conjugate axis 5 , $e = \frac{\sqrt{61}}{6}$, Latus rectum = $\frac{25}{6}$, $\left(\pm\frac{\sqrt{61}}{2}, 0\right)$