LESSON 10

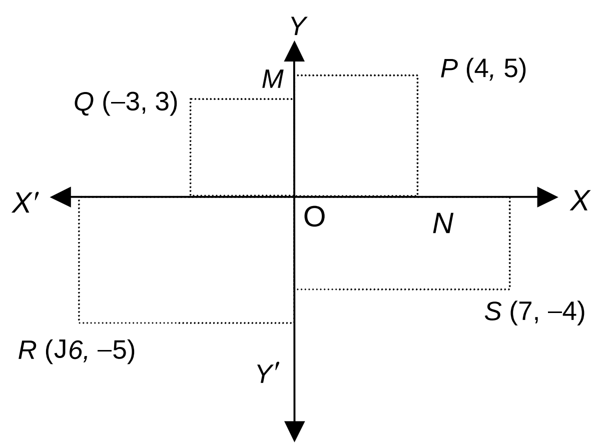
STRAIGHT LINES

1. PRELIMINARIES

The system of Geometry in which a point is specified by means of an ordered number-pair is known as Coordinate Geometry. It enables us to solve geometrical problems by algebraic methods.

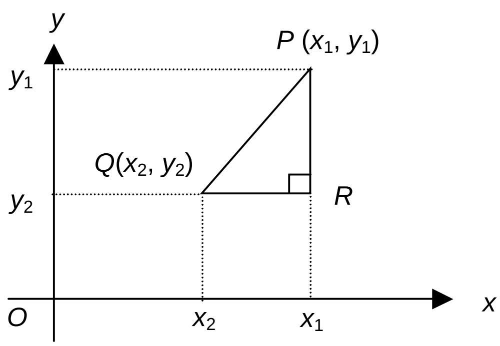
1.1 COORDINATES OF POINT

Let P be a point in a plane; draw two perpendicular lines $X'OX, Y'OY$ in the plane, and draw PM, PN parallel to OX, OY respectively. $X'OX, Y'OY$ are known as the coordinate axes. MP is the x-coordinate, called the abscissa, and NP is the y-coordinate called the ordinate of P. These two distances MP and NP together fix the position of P in the plane.



If $MP = 4$, $NP = 5$, then the position of P is denoted by (4, 5) the x-coordinate being written first. Q is the point (–3, 3), for the x-coordinate, is measured to the left of the y-axis, and is negative in sign; R is the point (–6, –5) for both coordinates are negative in sign. Similarly S is the point (7, –4). It should be observed that *O*, the origin is the point (0, 0) and the *y*-coordinate of any point on the *x*-axis is zero while the *x*-coordinate of any point on the *y-axis* is zero.

1.2 TO FIND THE DISTANCE BETWEEN TWO GIVEN POINTS IN THE CARTESIAN SYSTEM



Let the coordinates of the given points *P* and *Q* be (x_1, y_1) and (x_2, y_2) respectively.

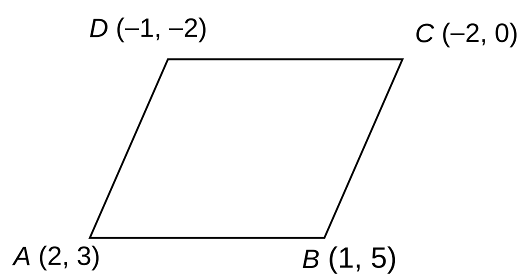
From the Figure

$$PQ = \sqrt{QR^2 + PR^2} \quad i.e., \quad \mathbf{PQ} = \sqrt{(\mathbf{x_1} - \mathbf{x_2})^2 + (\mathbf{y_1} - \mathbf{y_2})^2}$$

Illustration 1

Question: **Show that the points (2, 3) (1, 5) (J2, 0) and (J1, J2) are vertices of a parallelogram.**

Solution: Let the points be denoted by *A, B, C, D* in order



$AB^2 = (2-1)^2 + (3-5)^2 = 1 + 4 = 5$

$BC^2 = (1+2)^2 + (5-0)^2 = 9 + 25 = 34$

$CD^2 = (-2+1)^2 + (0+2)^2 = 1 + 4 = 5$

$DA^2 = (-1-2)^2 + (-2-3)^2 = 9 + 25 = 34$

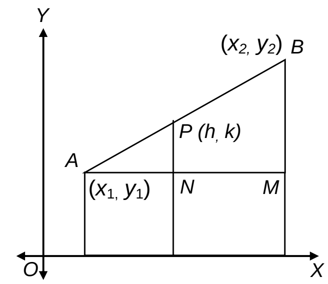
Since the opposite sides are equal, the figure drawn is a parallelogram.

2. SECTION FORMULA

Let A, B are the points (x_1, y_1), (x_2, y_2), here we will find the coordinates of the point P on AB such that $AP : PB = l : m$.

Let the coordinates of P be (h, k)

In the Figure, triangles ANP and AMB are similar.



$$\therefore \quad \frac{PN}{BM} = \frac{AP}{AB} = \frac{l}{l+m}, \text{ since } \frac{AP}{PB} = \frac{l}{m} \quad i.e., \frac{k - y_1}{y_2 - y_1} = \frac{l}{l+m}$$

$$\therefore \quad (l+m)k = (l+m)y_1 + l(y_2 - y_1) = ly_2 + my_1$$

$$\therefore \quad k = \frac{ly_2 + my_1}{l+m}$$

similarly, $h = \dfrac{lx_2 + mx_1}{l+m}$

Note: Putting $l = m$, we find that the mid-point of the line joining the points $(x_1, y_1), (x_2, y_2)$, i.e. $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Note : When P lies outside AB i.e., external to AB such that $AP : BP = l : m$.

We have h = $\dfrac{lx_2 - mx_1}{l-m}, k = \dfrac{ly_2 - my_1}{l-m}$

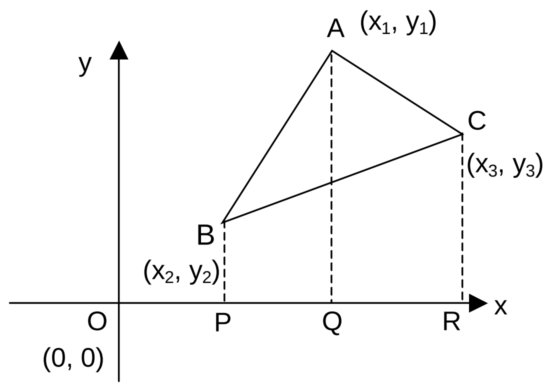
It has been assumed that l and m are positive numbers; if, however, we take $AP : BP$ to be positive in the former case and negative in the latter case, we have in both cases the formula

$$\boldsymbol{h = \frac{lx_2 + mx_1}{l+m}, k = \frac{ly_2 + my_1}{l+m}}$$

3. AREA OF A TRIANGLE

Let the vertices of the triangle be
$A(x_1, y_1),\ B(x_2, y_2),\ C(x_3, y_3)$

We have considered the vertices of the triangle in the first quadrant for the ease of calculations. Drop perpendiculars form *A, B* and *C* on the *x*-axis.



Area of ΔABC = Area of trapezium *ABPQ*

+ Area of trapezium *AQRC*

– Area of trapezium *BCRP*

$$= \frac{1}{2}(BP + AQ)(PQ) + \frac{1}{2}(AQ + CR)(QR) - \frac{1}{2}(BP + CR)(PR)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

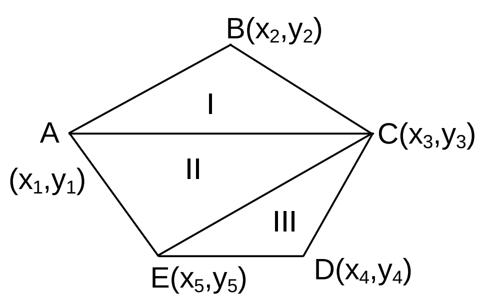
expressed in the determinant form,

$$\Rightarrow \text{ a } \textbf{= absolute value of } \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

The area of a triangle is considered positive when its vertices are named in counter-clockwise order, and negative when named in clockwise order.

If three points are collinear, we have the area of the Δ = 0.

Similarly to calculate the area of pentagon whose vertices are given, we can divide the pentagon into a number of triangles and then sum up the areas. e.q.



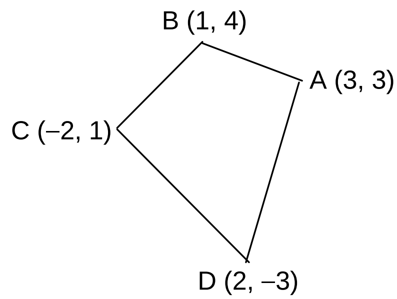
Area of pentagon *ABCDE* = Area of I + Area of II + Area of III

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}$$

Illustration 2

Question: **Find the area of the quadrilateral with vertices (3, 3), (1, 4), (J2, 1), (2, J3).**

Solution: Let the given vertices be *A, B, C, D* respectively.



Join *AC*

Area of *ABCD* = area of triangle *ABC* + area of triangle *ACD*

$$= \frac{1}{2}\left[3(4-1)+1(1-3)-2(3-4)\right]+\frac{1}{2}\left[3(1+3)-2(-3-3)+2(3-1)\right]$$

$$= \frac{1}{2}(9-2+2+12+12+4) = \frac{37}{2}.$$

4. EQUATION OF LOCUS OF A POINT

The locus of a point, as it moves in accordance with a given geometrical condition, is the path traced out by the moving point. In geometry we mean, by locus, the curve itself. We say that the locus is a straight line (or) the locus is a circle. On the contrary in coordinate geometry the moving point can be represented by the ordered pair (*x, y*). The geometrical condition imposed on the moving point gets transformed into an algebraic relation connecting (*x, y*). This relation, between *x* and *y* (the coordinates of the moving point), is called the equation to the curve described by the points.

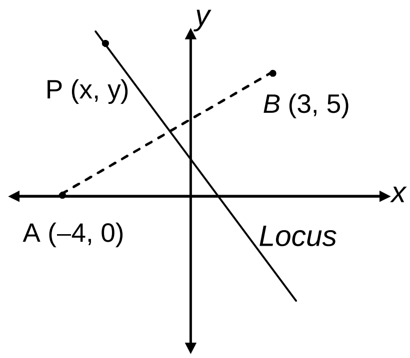
For example, if *P* (*x, y*) is any point equidistant from *A (J4, 0) and B* (3, 5),

we have $PA^2 = PB^2$ and this can be expressed in terms of coordinates of *A, P* and *B* as

$$(x+4)^2 + y^2 = (x-3)^2 + (y-5)^2$$

i.e., $14x + 10y - 18 = 0$

i.e., $7x + 5y - 9 = 0$



Moreover, if P (x, y) is any point whose coordinates satisfy the equation $7x + 5y - 9 = 0$,

We have $PA^2 = PB^2$

i.e., P is equidistant from *A* and *B*. Hence the relation $7x + 5y - 9 = 0$, is called the equation of the locus of points equidistant from *A* and *B.*

It is said that the equation $7x + 5y - 9 = 0$, represents the perpendicular bisector of *AB.*

Coordinate Geometry is based on the concept of locus of a point. Large number of problems will involve the idea of the locus of a point.

To find locus of a point *P,* proceed as follows:

1. Assume co-ordinate of *P* as (h, k); h and k are unknown quantities.
2. Translate the given problem into equations, which will contain coordinates of *P,* some given quantities and some unknown quantities.
3. Suppose two unknown quantities are introduced in the above method. Then one must get three equations (or more) in step 2.
4. Eliminate unknowns to get an equation which should be generalized by replacing h by x, k by y, to get the required locus.

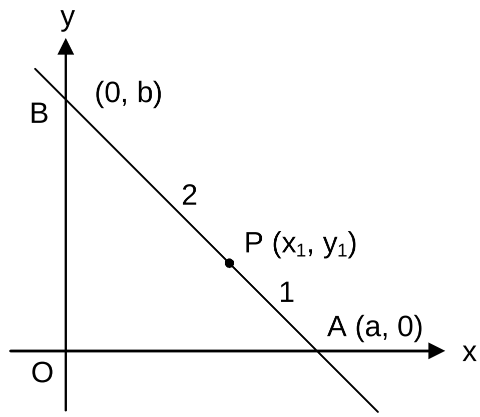
Illustration 3

Question: **A straight line segment of length *'p'* moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line in the ratio 1 : 2.**

Solution: Choose the two mutually perpendicular lines as axes of coordinates.

The straight line segment *AB* of constant length '*p'* slides so that *A* and *B* move along *OX* and *OY* respectively. Let P (x_1, y_1) be a point of *AB* such that

$AP : PB = 1 : 2$. It is required to find the locus of *P.*



At any position of *AB,* let the intercepts *OA, OB* be *a, b* respectively, so that

$a^2 + b^2 = p^2$ ...(i)

Since *AP : PB* = 1 : 2, we have

$x_1 = \frac{2a}{3}, y_1 = \frac{b}{3}$

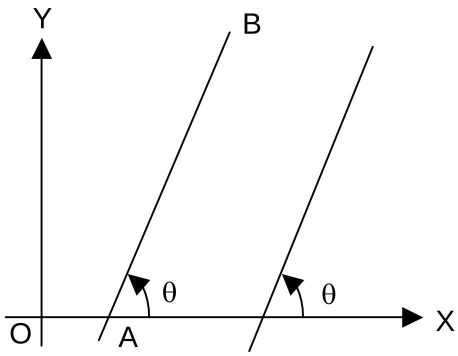
$\therefore$ a = $\frac{3x_1}{2}, b = 3y_1$

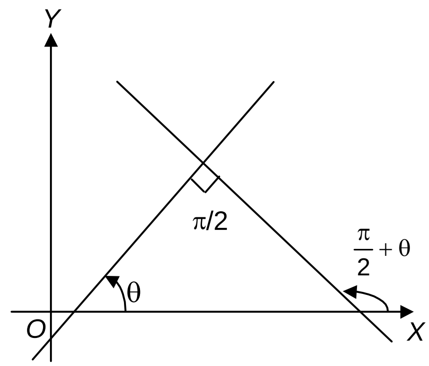
Using (i) $\left(\frac{3x_1}{2}\right)^2 + (3y_1)^2 = p^2$, and this implies that

$P(x_1, y_1)$ should lie on the curve $\left(\frac{3x}{2}\right)^2 + (3y)^2 = p^2$

i.e., the required locus is $\frac{x^2}{4} + \frac{y^2}{1} = \frac{p^2}{9}$.

5. SLOPE OF A LINE





If a straight line *AB* makes an angle θ with the positive direction of the *x*-axis, then $\tan\theta$ is called the slope or gradient of the straight line and is usually denoted by the letter '*m*'.

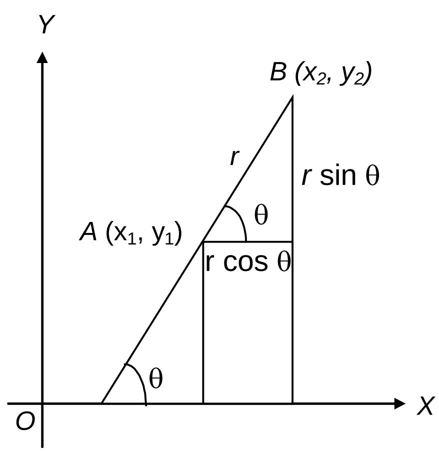
It follows that

(i) if two lines are parallel, their slopes are equal, for the lines must be equally inclined to the positive direction of the *x*-axis.

(ii) if two lines are perpendicular the product of their slopes is -1, for if one line is inclined at an angle θ to the *x*-axis, the other must be inclined at $\frac{\pi}{2}+\theta,$ hence their slopes are $\tan\theta$ and $\tan\left(\frac{\pi}{2}+\theta\right)$, i.e., $\tan\theta$ and $-\cot\theta$.

$\therefore$ the product is -1.

5.1 TO FIND THE SLOPE OF THE LINE JOINING ANY TWO POINTS (x_1, y_1) AND (x_2, y_2)



Let the segment joining of the points be of length *r* and let the line be inclined to the positive direction of *x*-axis at angle θ.

$\therefore \quad r\cos\theta = x_2 - x_1$

$r \sin\theta = y_2 - y_1$

and therefore $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

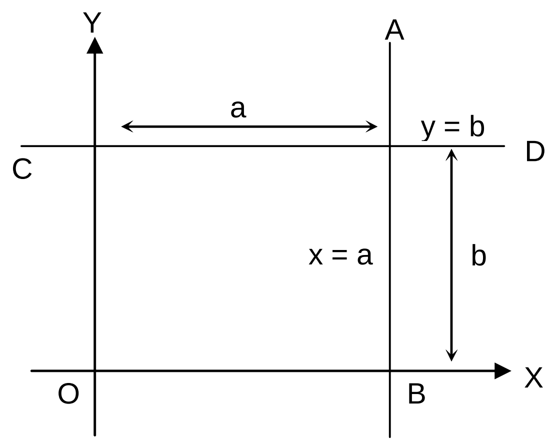
and $\tan\theta$ is the required slope. The expression for the slope is, therefore,

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{difference of the ordinates of the two points}}{\text{difference of the abscissa}} \text{ provided } x_1 \neq x_2$$

If $x_1 = x_2$, then m is not defined therefore given line is perpendicular to x-axis.

5.2 LINES PARALLEL TO THE CO ORDINATE AXES

Let the line *AB* be parallel to the *Y*-axis and at a distance '*a*' from it. Every point on *AB* will have its abscissa 'a', and hence the equation of *AB* is $x = a$. By putting $a = 0$, we deduce that the equation of the *y*-axis is $x = 0$.



Similarly, the equation of the straight line *CD* parallel to the *X*-axis and at a distance '*b*' from it is $y = b$. By putting $b = 0$ we deduce that the equation to the *X*-axis is $y = 0$

5.3 ANGLE BETWEEN TWO GIVEN STRAIGHT LINES

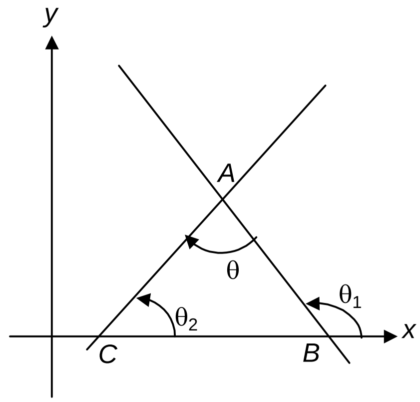
Let *AB, AC* have slopes m_1, m_2 and be inclined to the *X*-axis at θ_1, θ_2.

Then $\angle = \theta_1 - \theta_2 = \theta$

$$\therefore \tan \angle CAB = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

i.e., $\tan^{-} = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

where θ is the angle between *AB* and *AC*



Note: It is customary to take θ, as the acute angle between the two lines, and hence mostly one can take the above formula as tan $\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

If the lines are parallel,

tan θ = 0, since θ = 0

$\therefore\ m_1 - m_2 = 0$

$\therefore\ \boldsymbol{m_1 = m_2}$

and **if the lines are perpendicular,** tan θ is not defined, since $\theta = \frac{\pi}{2}$, and

therefore $m_1 m_2 + 1 = 0$

$\therefore\ \boldsymbol{m_1 m_2 = -1}$.

Illustration 4

Question: **Find the acute angle between the two lines with slopes $\frac{1}{5}$ and $\frac{3}{2}$.**

Solution: If the angle between the lines is θ,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{5} - \frac{3}{2}}{1 + \left(\frac{1}{5}\right) x \left(\frac{3}{2}\right)}\right| = |-1| = 1$$

Therefore the angle is 45°.

6. VARIOUS FORMS OF THE EQUATION OF A STRAIGHT LINE

It was said earlier that every curve, in coordinate geometry, is represented by an equation which is a relation connecting x and y, the coordinates of any point which lies on that curve. In particular, any straight line has also an equation to represent it. It may be seen that to represent a straight line we need to be given two independent information; and depending upon the two, the equation also changes.

We have therefore the following forms of equations to straight lines:

6.1 SLOPE ONE POINT FORM

Given that *a* line has a slope (*m = tan* ¯) which gives the direction it may be noted that '*m*' alone does not give the equation of the line and with the same slope there can be any number of straight lines all of which are parallel. Given that it passes through a given point (x_1, y_1). In this case the equation has the form.

$\boldsymbol{y \text{ J } y_1 = m(x \text{ J } x_1)}$

Illustration 5

Question: **If a line has a slope = $\frac{1}{2}$ and passes through *(*J *1, 2);* find its equation.**

Solution: The equation of the line is

$$y-2=\frac{1}{2}(x-(-1))$$

i.e., $2y-4=x+1$

i.e., $x-2y+5=0$

6.2 Y-INTERCEPT FORM

Given the slope '*m*' and the length '*c*' (called the intercept) cut off on the *y*-axis by the line. In this case the form of the equation is

$\boldsymbol{y = mx + c}$

Illustration 6

Question: **If a line has a slope $\frac{1}{2}$ and cuts off along the positive *y*-axis of length $\frac{5}{2}$ find the equation of the line.**

Solution: $y=\frac{1}{2}x+\frac{5}{2}$

i.e., $2y=x+5$

i.e., x J $2y+5=0$

6.3 TWO POINT FORM

Equation of a line passing through two points (x_1, y_1) and (x_2, y_2).

In this case the equation to the line is of the form

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Where $x_1 \neq x_2$ and $\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$ is the slope (m) of the line.

Illustration 7

Question: **If a line passes through two points (1, 5) and (3, 7), find its equation.**

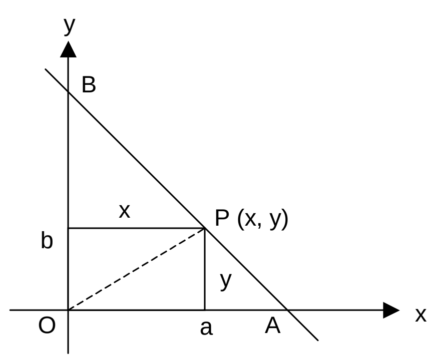
Solution: $\frac{y-5}{x-1} = \frac{-7+5}{-3+1} = \frac{-2}{-2} = 1$ *i.e.,* $y - 5 = x - 1$

i.e., x J *y* + 4 = *0*

6.4 INTERCEPT FORM

Let the line make intercepts *a* and *b* on the axes of *x* and *y* respectively.

Let *P*(*x, y)* represent any point on the line. Join *OP.*



We have,

Area of triangle *OAB* = area of triangle *OAP* + area of triangle *OPB*

i.e., $\frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$

or $\frac{x}{a} + \frac{y}{b} = 1$

This is called the intercept form of the equation of straight line.

Illustration 8

Question: **Find the equation of the straight line, which passes through the point (3, 4) and whose intercept on *y*-axis is twice that on *x*-axis.**

Solution: Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad \ldots(i)$$

According to the question $b = 2a$

$\therefore$ from (i) equation of line will become

$$\frac{x}{a}+\frac{y}{2a}=1 \qquad \text{or } 2x + y = 2a \quad \ldots.(ii)$$

Since line (ii) passes through the point (3, 4)

$\therefore\ 2 \times 3 + 4 = 2a \qquad \therefore\ a = 5$

$\therefore$ from (ii), equation of required line will be $2x + y = 10$.

6.5 NORMAL FORM

If p is the length of the perpendicular from the origin upon a straight line, and that α is the angle the perpendicular makes with the axis of x, equation of the straight line can be obtained as follows:

Let PQ be the straight line, OQ (= p) the perpendicular drawn to it from the origin O, and $\angle QOX = \alpha$

Draw the ordinate PN also draw NR perpendicular to OQ, and PM perpendicular to RN

We have $\angle PNM = 90^\circ - \angle RNO = \alpha$

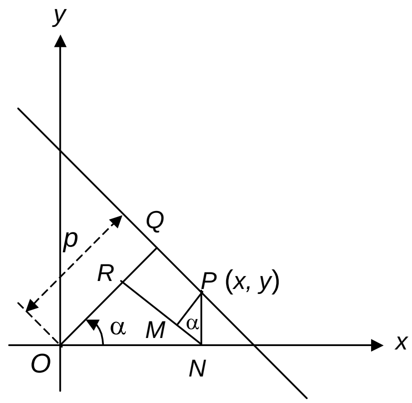
$\therefore\ p = OQ = OR + RQ = OR + PM = ON\ \cos\ \alpha + PN\ \sin\ \alpha$ or $p = x\cos\alpha + y\sin\ \alpha$

$\therefore\ x\cos\alpha + y\sin\alpha = p$ is the required equation.

This is called the perpendicular form.

Alternatively, suppose $\frac{x}{a}+\frac{y}{b}=1$ is the equation of the straight line

In this case, $a=\frac{p}{\cos\alpha}, b=\frac{p}{\sin\alpha}$



$\therefore$ by substitution, the equation becomes

$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1 \text{ or } \boldsymbol{x\cos\alpha + y\sin\alpha = p}$$

This is known as the normal form of the equation to a straight line.

Illustration 9

Question: **Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of *x*-axis.**

Solution: Let *AB* be the required line and *OL* be perpendicular to it.

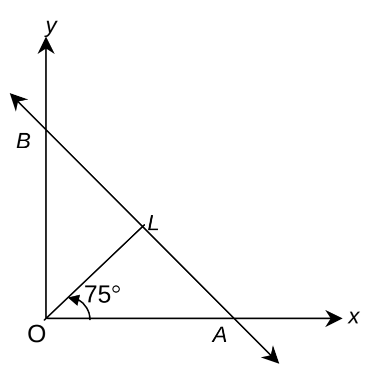
Given $OL = 3\sqrt{2}$ and $\angle LOA = 75°$

$\therefore$ equation of line *AB* will be

$x \cos 75° + y \sin 75° = 3\sqrt{2}$

[Normal form] ...(i)

Now $\cos 75° = \cos (30° + 45°) = \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$



and $\sin 75° = \sin (30° + 45°) = \sin30° \cos45° + \cos30° \sin45° = \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore$ from (i) equation of line *AB* is

$$x\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+y\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=3\sqrt{2} \text{ or } \left(\sqrt{3}-1\right)x+\left(\sqrt{3}+1\right)y-12=0$$

6.6 GENERAL FORM

Any linear (1st degree in x and y) equation of the form

$Ax + By + C = 0$

represents a straight line.

The straight line in this form, has

(a) slope = $m = -\dfrac{A}{B} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

(b) x intercept = $-\dfrac{C}{A}$ and y intercept = $-\dfrac{C}{B}$

Illustration 10

Question: **Find the value of k so that the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ and $7x + 5y - 4 = 0$ are perpendicular to each other.**

Solution: Given lines are

$(2 + 6k)\, x + (3 - k)\, y + 4 + 12k = 0$ …(i)

and $7x + 5y - 4 = 0$ …(ii)

slope of line (i), $m_1 = -\dfrac{2+6k}{3-k} = \dfrac{2+6k}{k-3}$

and slope of line (ii), $m_2 = -\dfrac{7}{5}$

Since line (i) is perpendicular to line (ii)

$\therefore \left(\dfrac{2+6k}{k-3}\right)\left(-\dfrac{7}{5}\right) = -1$

or $(2 + 6k)\, 7 = 5\,(k - 3)$

or $14 + 42k = 5k - 15$

or $37k = -29$

or $k = -\dfrac{29}{37}$

7. POINT OF INTERSECTION OF TWO LINES

Consider two non-parallel lines : $ax + by + c = 0$ and $a'x + b'y + c' = 0$

$$i.e. -\frac{a}{b} \neq -\frac{a'}{b'} \Rightarrow ab' - a'b \neq 0$$

Their point of intersection can be obtained by solving their equations simultaneously which can be done through substitution, elimination or cross-multiplication rule. The point of intersection is given by

$$x = \frac{bc' - b'c}{ab' - a'b}, \quad y = \frac{a'c - ac'}{ab' - a'b}$$

Illustration 11

Question: **Show that the lines 2*x* – *y* – 12 = 0 and 3*x* + *y* – 8 = 0 intersect at a point which is equidistant from both the coordinate areas.**

Solution: $2x - y - 12 = 0$

$3x + y - 8 = 0$

Using cross multiplication rule, we have

$$\frac{x}{(-1)(-8)-(1)(-12)} = \frac{-y}{(2)(-8)-(3)(-12)} = \frac{1}{(2)(1)-(3)(-1)} \Rightarrow \frac{x}{20} = \frac{-y}{20} = \frac{1}{5}$$

$$\Rightarrow x = \frac{20}{5} = 4 \text{ and } y = -\frac{20}{5} = -4$$

So, the point is (4, –4) which is at a distance 4 units from both the coordinate axes.

8. AREA OF TRIANGLE WHEN EQUATION OF SIDES ARE GIVEN

Let equations of sides be

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\therefore$ area of triangle $ABC = \dfrac{\Delta^2}{2|C_1C_2C_3|}$

where C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 in the determinant Δ.

Illustration 12

Question: **Find the area of triangle formed by the lines *x* – *y* + 1 = 0, 2*x* + *y* + 4 = 0 and *x* + 3 = 0.**

Solution: Area $= \dfrac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}^2$

where $C_1 = (2)\,(0) - (1)\,(1) = -1$

$C_2 = -[(1)\,(0) - (1)\,(-1) = -1$

$C_3 = (1)\,(1) - (2)\,(-1) = 3$

Putting these values, we get

Area $= \dfrac{1}{2(3)}\left[1(-4-1)+3(1+2)\right]^2$ (expanding along last row)

$= \dfrac{8}{3}$ sq. units

9. CONDITION OF COLLINEARITY

Three points *A, B* and *C* or more are said to be collinear when they lie on the same line. Collinearity can be checked in two simple ways.

(i) If Slope of *AB* = slope of *BC*, points *A, B, C* are collinear.

(ii) If Area of $\Delta ABC = 0$, points *A, B, C* are collinear.

(iii) Collinear points must satisfy section formulae.

Illustration 13

Question: **Prove that the points (1, 2), (-3, a) and (4, 0) are collinear if a is equal to $\dfrac{14}{3}$.**

Solution: Let *A* (1,2), *B* (-3, λ) and *C* (4, 0) to be the three points.

Slope of *AB* = slope of *AC*

$\Rightarrow \dfrac{\lambda-2}{-3-1} = \dfrac{2-0}{1-4} \Rightarrow \dfrac{\lambda-2}{-4} = -\dfrac{2}{3}$

$\Rightarrow \lambda = 2 + \dfrac{8}{3} = \dfrac{14}{3}$

Alternatively,

Area (Δ *ABC*) $= \dfrac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -3 & \lambda & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow\ 4(2-\lambda)+(\lambda+6)=0$

$\Rightarrow\ -4\lambda+\lambda+8+6=0$

$\Rightarrow\ \lambda=\dfrac{14}{3}$

10. CONDITION OF CONCURRENCY

Three or more lines are said to be concurrent if they all pass through the same point.

If the lines

$a_1 x + b_1 y + c_1 = 0$

$a_2 x + b_2 y + c_2 = 0$

$a_3 x + b_3 y + c_3 = 0$

are concurrent, the area of triangle formed by them is zero.

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Illustration 14

Question: **The line $x + {}^{a}y - 4 = 0$ passes through the point of intersection of $4x - y + 1 = 0$ and $x + y + 1 = 0$. Find the values of ^{a}.**

Solution: The three lines are concurrent

$$\Rightarrow \begin{vmatrix} 1 & \lambda & -4 \\ 4 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \ \Rightarrow\ -2-3\lambda-20=0 \ \Rightarrow\ \lambda = -\frac{22}{3}$$

11. FAMILY OF LINES

Family means a group of lines having a particular characteristic. e.g., family of lines parallel to x-axis are of the form $y = k$, where k is any arbitrary real number. Family of lines perpendicular to $2x - y + 4 = 0$ is $x + 2y + k = 0$ since product of slopes has to be–1. so, we can say

(i) Family of lines parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$

(ii) Family of lines perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$

(iii) Family of lines through the point of intersection of the lines

$ax + by + c = 0$ ..(i)

and $a'x + b'y + c' = 0$ ...(ii)

can be written as

$(ax + by + c) + \text{k}\,(a'x + b'y + c') = 0$ ...(iii)

Equation (i) represents a straight line, for it is of the first degree.

Let (x_1, y_1) be the coordinates of the common point of (i) and (ii)

The point $(x_1, ,y_1)$ lies on (i);

$\therefore\ ax_1 + by_1 + c = 0$ ...(iv)

The point $(x_1, ,y_1)$ lies on (ii)

$\therefore\ a'\,\text{x}_1 + b'\,y_1 + c' = 0$ …(v)

Multiplying (v) by k, and adding to (iv), we get

$ax_1 + by_1 + c + k\left(a'\,x_1 + b'\,y_1 + c'\right) = 0$ ...(vi)

From this we see that the values x_1, y_1 satisfy the equation (iii)

$\therefore$ the straight line represented by (iii) passes through $(x_1, ,y_1)$ the common point of (i) and (ii).

Illustration 15

Question: **Find the equation of a line parallel to $x + 2y = 3$ and passing through the point (3, 4).**

Solution: Any line parallel to $x + 2y = 3$ is of the form $x + 2y = k$.

The point (3,4) satisfies this equation

$\Rightarrow 3 + 2(4) = k \Rightarrow k = 11$

So, required line is $x + 2y = 11$

Illustration 16

Question: **Find the equation of line perpendicular to $2x - 3y = 5$ and cutting off an intercept 1 on the x-axis.**

Solution: Any line perpendicular to $2x - 3y = 5$ is of the form $3x + 2y = k$

Putting $y = 0$, we get $x = \frac{k}{3}$, the x-intercept.

Since, $\frac{k}{3} = 1 \Rightarrow k = 3$

The required line is $3x + 2y - 3 = 0$

Illustration 17

Question: **Find the equation of the straight line passing through (2, J9) and the point of intersection of lines 2*x* + 5*y* J 8 = 0, 3*x* J4*y* J35 = 0.**

Solution: Any line through the point of intersection of given lines is $2x + 5y - 8 + K\,(3x - 4y - 35) = 0$

Point (2, –9) lies on it, implies $K = 7$.

$\therefore$ The required line becomes x J y – 11 = 0

Illustration 18

Question: **Find the equation of straight line passing through the point of intersection of lines 3*x* J 4*y* + 1 = 0, 5*x* + *y* J 1 = 0 and cutting off equal intercepts from coordinate axes.**

Solution: Any line passing through the point of intersection of given lines is

$$3x - 4y + 1 + K(5x + y - 1) = 0$$

or $(3 + 5K)x + (K - 4)y = K - 1$

Writing it in intercept form as

$$\frac{x}{\frac{K-1}{3+5K}} + \frac{y}{\frac{K-1}{K-4}} = 1, \text{ if } K \neq 1$$

For equal intercepts: $\frac{K-1}{K-4} = \frac{K-1}{3+5K}$ which gives $K = \frac{-7}{4}$

Hence the required equation is $23x + 23y = 11$

12. DISTANCE OF A POINT FROM A LINE

The perpendicular distance of a point from a line can be obtained when the equation of the line and the coordinates of the point are given.

Case I :

Let us first derive the formula for this purpose when the equation of the line is given in normal form.

Let the equation of the line l in normal form be

$x \cos \alpha + y \sin \alpha = p$,

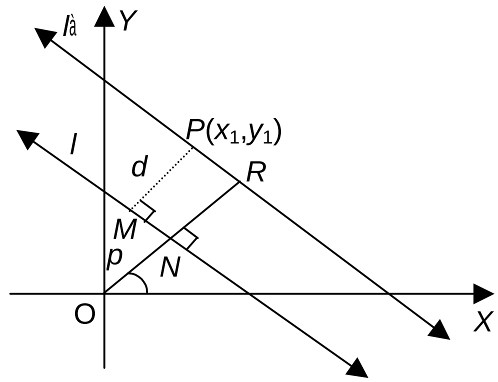
when α is the angle made by the perpendicular from origin to the line with positive direction of x-axis and p is the length of this perpendicular. Let $P(x_1, y_1)$ be the point, not on the line l. Let the perpendicular drawn from the point P to the line l be PM and $PM = d$. Point P is assumed to lie on opposite side of the line l from the origin O. Draw a line l' parallel to the line l through the point P. Let ON be perpendicular from the origin to the line l which meets the line l' in point R.

Obviously

$ON = p$ and $\angle XON = \alpha$

Also, from the Figure, we note that

$OR = ON + NR = p + MP$



Therefore, length of the perpendicular from the origin to the line l' is

$OR = p + d$

and the angle made by the perpendicular OR with positive direction of x-axis is α. Hence, the equation of the line l' in normal form is

$x \cos \alpha + y \sin \alpha = p + d$

Since the line l' passes through the point P, the coordinates (x_1, y_1) of the point P should satisfy the equation of the line l' giving

$x_1 \cos \alpha + y_1 \sin \alpha = p + d$,

or $\quad d = x_1 \cos \alpha + y_1 \sin \alpha - p$.

The length of a segment is always non–negative. Therefore, we take the absolute value of the RHS, i.e.,

$d = | x_1 \cos \alpha + y_1 \sin \alpha - p |$.

Thus, the length of the perpendicular is the absolute value of the result obtained by substituting the coordinates of point P in the expression $x \cos \alpha + y \sin \alpha - p$

Case II:

Let the equation of the line be

$Ax + By + C = 0$

Reducing the general equation to the normal form, we have

$$\pm \frac{A}{\sqrt{A^2 + B^2}} x \pm \frac{B}{\sqrt{A^2 + B^2}} y = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

Where sign is taken + or – so that the RHS is positive.

(a) When $C < 0$

In this case the normal form of the equation of the line / becomes

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}$$

Now, from the result of case (I), the length of the perpendicular segment drawn from the point $P(x_1, y_1)$ to the line (II) is

$$d=\left|\frac{A}{\sqrt{A^2+B^2}}x_1+\frac{B}{\sqrt{A^2+B^2}}y_1+\frac{C}{\sqrt{A^2+B^2}}\right|$$

i.e., $$d=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

(b) When $C > 0$

In this case the normal form of the equation (II) becomes

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

Again, by the result of case (I), the distance d is

$$d=\left|\frac{-Ax_1-By_1-C}{\sqrt{A^2+B^2}}\right|$$

or $$d=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

The perpendicular distance of (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

So, Perpendicular distance from origin is $\frac{|c|}{\sqrt{a^2+b^2}}$.

Illustration 19

Question: **The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. Find the length of side of the triangle.**

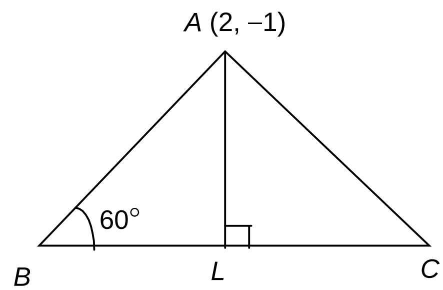
Solution: Equation of side BC is

$x + y - 2 = 0$ …(i)

$A \equiv (2, -1)$

Now $AL = \left|\dfrac{2-1-2}{\sqrt{1^2+1^2}}\right| = \dfrac{1}{\sqrt{2}}$

From ΔABL, $\sin 60^\circ = \dfrac{AL}{AB}$; $\therefore \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{2}AB}$ or $AB = \sqrt{\dfrac{2}{3}}$



$x + y - 2 = 0$

13. DISTANCE BETWEEN PARALLEL LINES

The distance between $ax + by + c = 0$ and $ax + by + c' = 0$

is given by, $d = \dfrac{|c - c'|}{\sqrt{a^2 + b^2}}$

Make sure that coefficients of x and y are same in both the lines before applying the formula.

Illustration 20

Question: **Find the distance between the lines $5x + 12y + 40 = 0$ and $10x + 24y - 25 = 0$.**

***Solution*:** Here, coefficients of x and y are not the same in both the equations. So, we write them as

$5x + 12y + 40 = 0$

$5x + 12y - \dfrac{25}{2} = 0$

Now, distance between them = $\dfrac{\left|40 - \left(-\dfrac{25}{2}\right)\right|}{\sqrt{(5)^2 + (12)^2}} = \dfrac{105}{2(13)} = \dfrac{105}{26}$

PRACTICE PROBLEMS

PP1. Let A (6, -1), B (1, 3) and C (λ, 8) be three points such that AB = BC, find the value (s) of λ.

PP2. Find the equation of line joining the points (–1, 3) and (4, -2).

PP3. Find the equation of line whose intercept on x-axis and y-axis are respectively twice and thrice of those by the line $3x + 4y = 12$.

PP4. Find the y – intercept of the line $3x - 2y + 7 = 0$.

PP5. Find the acute angle between the lines $y = 4x - 1$ and $x = 4y + 1$.

PP6. Find the area of triangle formed by the lines $x + y + 1 = 0$, $x = 5$ and $y = -3$.

PP7. Find the equation line parallel to $x - 2y + 4 = 0$ and at a distance 4 from it.

PP8. Find the distance of $3x - 4y + 7 = 0$ from origin.

PP9. Find the distance between the lines $3x - 4y + 7 = 0$ and $6x - 8y + 7 = 0$.

PP10. Find the equation line perpendicular to $2x + 5y - 1 = 0$ and passing through the point (4, 2).

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

If the algebraic sum of the perpendicular distances of a variable line from the points (0, 2), (2, 0) and (1, 1) is zero, then find the line always passes through the point.

Solution :

If the line be $ax + by + c = 0$, then $\frac{0+2b+c}{\sqrt{a^2+b^2}} + \frac{2a+c}{\sqrt{a^2+b^2}} + \frac{a+b+c}{\sqrt{a^2+b^2}} = 0$

$\Rightarrow$ $3a + 3b + 3c = 0$ $\Rightarrow$ $a + b + c = 0$

$\Rightarrow$ the line $ax + by + c = 0$ always passes through the point (1, 1).

***Example* 2:**

Find the equation of the locus of the points equidistant from (–1, –1) and (4, 2).

Solution :

Let $P(x, y)$ be any point on the locus and let the given points be $A(-1, -1)$ and $B(4, 2)$

Given $PA = PB \Rightarrow (PA)^2 = (PB)^2$

$\Rightarrow (x+1)^2 + (y+1)^2 = (x-4)^2 + (y-2)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = x^2 - 8x + 16 + y^2 - 4y + 4$

$\Rightarrow 10x + 6y - 18 = 0 \Rightarrow 5x + 3y - 9 = 0$ is the equation of required locus.

***Example* 3:**

The co-ordinates of points *P, Q, R* and *S* are (–3, 5), (4, –2), (*p*, 3*p*) and (6, 3) respectively and the areas of as ***PQR* and *QRS* are in ratio 2 : 3. Find *p*.**

Solution :

Area of $\Delta PQR = \frac{1}{2}|6 - 20 + 12p + 2p + 5p + 9p| = \frac{1}{2}|28p - 14|$

Area of $\Delta QRS = \frac{1}{2}|12p + 2p + 3p - 18p - 12 - 12|$

$$= \frac{1}{2}|-p-24| = \frac{1}{2}|-(p+24)| = \frac{1}{2}|p+24|$$

Given $\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta QRS} = \frac{2}{3}$

$$\Rightarrow \quad \frac{\left(\frac{1}{2}\right)|28p-14|}{\left(\frac{1}{2}\right)|p+24|} = \frac{2}{3} \quad \Rightarrow \quad \left|\frac{28p-14}{p+24}\right| = \frac{2}{3} \quad \Rightarrow \quad \frac{28p-14}{p+24} = \pm\frac{2}{3}$$

$$\Rightarrow \quad 84p - 42 = 2p + 48 \quad \text{or} \quad 84p - 42 = -2p - 48$$

$$\Rightarrow \quad 82p = 90 \quad \text{or} \quad 86p = -6$$

$$\Rightarrow \quad p = \frac{90}{82} \text{ or } p = \frac{-6}{86}$$

$$\Rightarrow \quad p = \frac{45}{41} \text{ or } p = \frac{-3}{43}$$

***Example* 4:**

Find the slope of the lines:

(a) passing through the points (3, –2) and (–1, 4)

(b) passing through the points (3, –2) and (7, –2)

(c) passing through the points (3, –2) and (3, 4)

(d) making inclination of 60° with the positive direction of *x*-axis

Solution :

(a) The slope of the line through (3, –2) and (–1, 4) is given by

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = \frac{-3}{2} \qquad \left[\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

(b) The slope of the line through the points (3, –2) and (7, –2) is given by

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0$$

(c) The slope of the line through the point (3, –2) and (3, 4) is given by

$m = \frac{4-(-2)}{3-3} = \frac{6}{0}$, which is not defined.

(d) Here inclination of the line i.e. $\theta = 60°$

$\therefore$ slope of the line is given by

$$m = \tan 60° = \sqrt{3}$$

***Example* 5:**

The points *A*(*a*, 0), *B*(0, *b*) and *C*(*x*, *y*) are collinear. Using slopes, prove that $\frac{x}{a} + \frac{y}{b} = 1$.

Solution :

The points *A, B, C* are collinear.

$\Rightarrow$ Slope of *AB* = Slope of *AC*

$$\Rightarrow \quad \frac{b-0}{0-a} = \frac{y-0}{x-a} \Rightarrow \frac{x-a}{-a} = \frac{y}{b}$$

$$\Rightarrow \quad \frac{x}{-a} + 1 = \frac{y}{b} \quad \Rightarrow \quad 1 = \frac{x}{a} + \frac{y}{b} \quad \Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

***Example* 6:**

If the angle between two lines is $\frac{®}{4}$ and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution :

We know that the acute angle θ between two lines having slope m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \ldots\text{(i)}$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$

Putting these values in (i), we get $\left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| = \tan\frac{\pi}{4} \Leftrightarrow \left|\frac{2m-1}{2+m}\right| = 1$

$$\Leftrightarrow \quad \left(\frac{2m-1}{2+m}\right) = \pm 1 \quad \Leftrightarrow \quad \frac{2m-1}{2+m} = 1 \text{ or } \frac{2m-1}{2+m} = -1$$

$$\Leftrightarrow \quad 2m - 1 = 2 + m \text{ or } 2m - 1 = -2 - m$$

$$\Leftrightarrow \quad m = 3 \text{ or } 3m = -1 \text{ i.e., } m = -\frac{1}{3}$$

Hence, the slope of the other line is 3 or $-\frac{1}{3}$

***Example* 7:**

Find the equation of the right bisector of the line joining (1, 1) and (3, 5).

Solution :

Slope of line joining (1, 1) and (3, 5) $= \frac{5-1}{3-1} = \frac{4}{2} = 2$ $\qquad \left[\because \text{ slope} = \frac{y_2 - y_1}{x_2 - x_1}\right]$

$\therefore$ Slope of right bisector of the join of (1, 1) and (3, 5) $= \left(\frac{-1}{2}\right)$

Also right bisector passes through mid-point of the join of (1, 1) and (3, 5) which is
$\left(\frac{1+3}{2}, \frac{1+5}{2}\right) = (2, 3)$

Thus the required right bisector passes through (2, 3) and has slope = $\left(-\frac{1}{2}\right)$ and so its equation is

$$y - 3 = \left(-\frac{1}{2}\right)(x - 2)$$

$\Rightarrow \quad 2y - 6 = -x + 2 \quad \Rightarrow \quad x + 2y - 8 = 0$

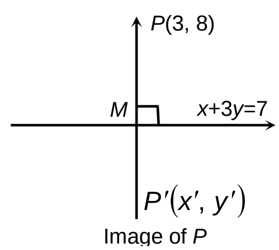
***Example* 8:**

(a) Show that the image of the points (3, 8) in the line $x + 3y = 7$ is $(-1, -4)$.

(b) Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution :

(a) Slope of given line $x + 3y = 7$ is $\left(-\frac{1}{3}\right)$ …(i)



$\therefore$ Slope of line *PM* perpendicular to (i) = 3

Equation of *PM* is $y - 8 = 3(x - 3)$

$\Rightarrow 3x - y - 1 = 0$ ….(ii)

Intersection of (i) and (ii) is obtained by solving i.e., *M*(1, 2).

Let $P'(x', y')$ be image of *P* in line (i)

$\therefore$ *M* is mid-point of PP'

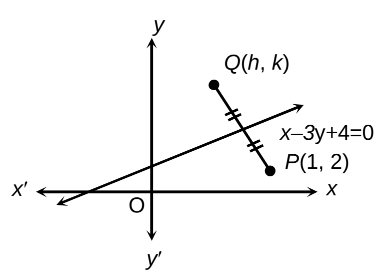
$\Rightarrow \quad (1, 2) = \left(\frac{3 + x'}{2}, \frac{8 + y'}{2}\right) \quad \Rightarrow \quad \frac{3 + x'}{2} = 1$ and $\frac{8 + y'}{2} = 2$

$\Rightarrow \quad x' = -1$ and $y' = -4$

$\Rightarrow \quad P'(-1, -4)$ is image of $P(3, 8)$ in line (i).

(b) Let *Q*(*h, k*) be the image of the point *P*(1, 2) in the line.

$x - 3y + 4 = 0$ …(i)



$\therefore$ The line (i) is the perpendicular bisector of line segment *PQ*.

Hence Slope of line *PQ* = $\dfrac{-1}{\text{slope of line } x - 3y + 4 = 0}$

$\Rightarrow \quad \dfrac{k-2}{h-1} = \dfrac{-1}{1/3}$ or $3h + k = 5$ …(ii)

and the mid-point of PQ i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (i) so that

$\dfrac{h+1}{2} - 3\left(\dfrac{k+2}{2}\right) + 4 = 0$ or $h - 3k = -3$ …(iii)

Solving (ii) and (iii), we get $h = \dfrac{6}{5}$ and $k = \dfrac{7}{5}$

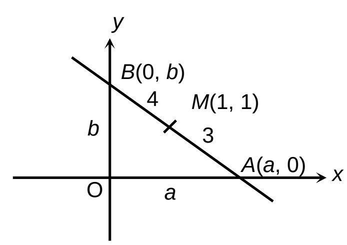
Hence, the image of the point (1, 2) in the line (i) is $\left(\dfrac{6}{5}, \dfrac{7}{5}\right)$

Example 9:

A straight line is such that the portion of it intercepted between the axes is bisected at the point (x_1, y_1). Prove that its equation is $\dfrac{x}{2x_1} + \dfrac{y}{2y_1} = 1$ or $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$.

Solution :

Let a and b be the intercepts made by the line on the axes.



$\therefore$ Its equation is $\dfrac{x}{a} + \dfrac{y}{b} = 1$ …(i)

Since $OA = a$ $\therefore$ A is $(a, 0)$

and $OB = b$ $\therefore$ B is $(0, b)$

Let $P(x_1, y_1)$ bisect the portion of the line between the axes i.e. AB.

Also by mid-point formula, P is $\left(\dfrac{a+0}{2}, \dfrac{0+b}{2}\right)$

Hence $x_1 = \dfrac{a}{2}$ and $y_1 = \dfrac{b}{2}$

$\Rightarrow \quad a = 2x_1$ and $b = 2y_1$ put in (i), we get $\dfrac{x}{2x_1} + \dfrac{y}{2y_1} = 1$ or $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$

Example 10:

Find the co-ordinates of a point on $x + y + 3 = 0$ whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.

Solution :

Let the required point be (h, k) on the line $x + y + 3 = 0$

$\Rightarrow \quad h + k + 3 = 0$ …(i)

Distance of (*h, k*) from the line $x + 2y + 2 = 0$ is $\sqrt{5}$

$\Rightarrow \quad \dfrac{|h + 2k + 2|}{\sqrt{1^2 + 2^2}} = \sqrt{5} \quad \Rightarrow \quad (h + 2k + 2) = \pm 5$

$\Rightarrow \; h + 2k - 3 = 0$ (taking +ve sign) …(ii)

$\Rightarrow \; h + 2k + 7 = 0$ (taking –ve sign) …(iii)

Solving (i) and (ii) we get $h = -9$ and $k = 6$

Solving (i) and (iii) we get $h = 1$ and $k = -4$

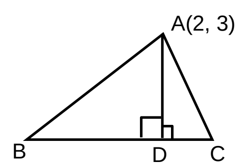
Hence the required points are $(-9, 6)$ and $(1, -4)$.

Example 11:

In the a **with vertices *A*(2, 3), *B*(4, –1) and *C*(1, 2), find the length of the altitude from the vertex *A*. Also find the equation of altitude.**

Solution :

Equation of *BC* is $y - (-1) = \dfrac{2 - (-1)}{1 - 4}(x - 4)$



$\Rightarrow \quad y + 1 = \dfrac{3}{-3}(x - 4) = -(x - 4)$

$\Rightarrow \quad x + y - 3 = 0$

$\therefore$ Length of altitude from vertex *A*

=perpendicular distance from *A*(2, 3) on the line $x + y - 3 = 0$

$= \dfrac{2 + 3 - 3}{\sqrt{1 + 1}} = \sqrt{2}$ units

Any line perpendicular to *BC* is $x - y + K = 0$

This passes through *A*(2, 3). $\therefore \; 2 - 3 + K = 0$, $\therefore \; K = 1$

$\therefore$ Equation of the altitude form *A* is $x - y + 1 = 0$

Example 12:

Find the value of *p* so that the three lines 3*x* H *y* J 2 Z 0 , *px* H 2*y* J 3 Z 0 and 2*x* J *y* J 3 Z 0 may intersect at one point.

Solution :

The equations of given lines are $3x + y - 2 = 0$ …(i)

$px + 2y - 3 = 0$ …(ii)

and $2x - y - 3 = 0$ …(iii)

Add (i) and (iii) to get point of intersection

$\therefore\ 5x-5=0 \ \Rightarrow\ x=1$ put in (i) and get $y=-1$

$\therefore$ (i) and (iii) meet in (1, –1)

This point $(1,-1)$ lies on (ii) because the given three lines meet in one point.

$\therefore\ p-2-3=0 \ \Rightarrow\ p=5$

***Example* 13:**

What are the points on the *x* axis whose perpendicular distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4?

Solution :

Let (*h*, 0) be the points on *x*-axis having perpendicular distance 4 from the line $\frac{x}{3}+\frac{y}{4}=1$ i.e.,

$4x+3y-12=0$

$$\Rightarrow\ 4=\frac{|4h+0-12|}{\sqrt{(4)^2+(3)^2}}=\frac{|4h-12|}{5} \ \Rightarrow\ 20=|4h-12|$$

$\Rightarrow\ 4h-12=\pm 20 \ \Rightarrow\ h=8$ or –2

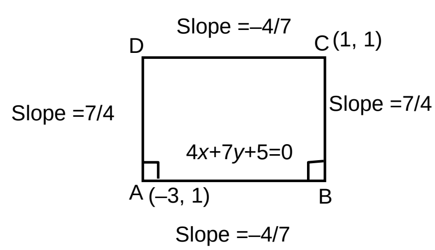
$\Rightarrow$ The required points are (*h*, 0) = (8, 0) and (–2, 0)

***Example* 14:**

One side of a rectangle lies along the line $4x+7y+5=0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Find the equation of other three sides.

Solution :

Let $4x+7y+5=0$ with slope $=\frac{-4}{7}$ be the side *AB* of rectangle *ABCD*.



Now $(-3, 1)$ satisfying $4x+7y+5=0$ and (1, 1) does not satisfy it.

Also line joining $(-3, 1)$ and $(1, 1)$ is not perpendicular to *AB* i.e., $4x+7y+5=0$, because product of their slopes is not –1.

$\therefore$ Let *A* be (–3, 1) and *C* be (1, 1)

Since *AD* and *BC* are perpendicular *AB*, so slopes of *AD* and *BC* = $-\frac{1}{(-4/7)}=\frac{7}{4}$

$\therefore$ Equation of *AD* is $y-1=\frac{7}{4}(x+3) \ \Rightarrow\ 4y-4=7x+21$

$\Rightarrow\ 7x-4y+25=0$

Equation of BC is $y-1=\frac{7}{4}(x-1) \Rightarrow 4y-4=7x-7$

$\Rightarrow \quad 7x-4y-3=0$

Equation of CD is $y-1=\frac{-4}{7}(x-1) \Rightarrow 7y-7=-4x+4$

$\Rightarrow \quad 4x+7y-11=0$

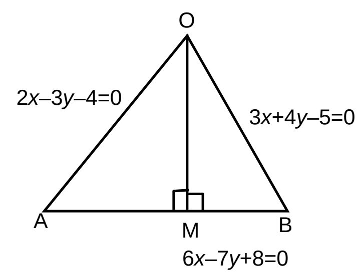
***Example* 15:**

A person standing at the junction (crossing) of two straight paths represented by the equations $\mathbf{2x}$ J $\mathbf{3y}$ J $\mathbf{4}$ Z $\mathbf{0}$ and $\mathbf{3x}$ H $\mathbf{4y}$ J $\mathbf{5}$ Z $\mathbf{0}$, wants to reach the path whose equation is $\mathbf{6x}$ J $\mathbf{7y}$ H $\mathbf{8}$ Z $\mathbf{0}$ in the least time. Find equation of the path that he should follow.

Solution :

Solving the equations of two given paths OA and OB i.e.,

$2x-3y-4=0$ and $3x+4y-5=0$ we have



$$\frac{x}{15+16}=\frac{y}{-12+10}=\frac{1}{8+9}$$

$$\Rightarrow \quad O(x,\, y)=0\left(\frac{31}{17},\, \frac{-2}{17}\right)$$

The person will reach the path AB i.e., $6x-7y+8=0$ in shortest time if the path followed is $OM \perp AB$.

$$\therefore \text{ Slope of } OM = \frac{-1}{\text{slope of } AB}=\frac{-1}{(-6/-7)}=\frac{-7}{6}$$

$$\therefore \quad \text{Equation of path } OM \text{ is } y-\left(\frac{-2}{17}\right)=\left(\frac{-7}{6}\right)\left(x-\frac{31}{17}\right) \Rightarrow \frac{17y+2}{17}=\frac{-7}{6}\left(\frac{17x-31}{17}\right)$$

$$\Rightarrow \quad 102y+12=-119x+217 \Rightarrow 119x+102y-205=0$$

EXERCISE – I

1. The mid points of the three sides of a triangle are (6, –1), (–1, –2) and (1, –4). Find the coordinates of the vertices of the triangle.

2. If the vertices of a triangle have integral co-ordinates, prove that the triangle cannot be equilateral.

3. Find the equation of the straight line perpendicular to the line $4x - 3y = 12$ and meeting it on the x-axis.

4. Find the value of k, such that the straight line $(2x + 3y + 4 = 0) + k\,(6x - y + 12 = 0)$ may be

(i) parallel to the y-axis

(ii) parallel to the line $7x + 5y - 4 = 0$

5. Find the equation of the line parallel to $5x - 12y = 10$, and at a distance $\frac{22}{13}$ from the point (1, 3).

6. Find the coordinates of the foot of the perpendicular drawn from the point (2, 3) on the line $y = 3x + 4$.

7. Transform the equation $x + \sqrt{3}y - 4 = 0$ to the normal form and hence find the length of perpendicular from the origin to the line.

8. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex.

9. If the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear, then find k.

10. Find y intercept of a line which passing through (2, 2) is perpendicular to the line $3x + y = 3$.

11. Let P and Q are the points on the line joining $A(-2, 5)$ ad $B(3, 1)$ such that $AP = PQ = QB$. Then find the mid-point of PQ.

12. If the line $6x - y + 2 + k(2x + 3y + 13) = 0$ is parallel to x-axis, then find the value of k.

EXERCISE – II

1. Find the equation of the straight line which is perpendicular to $y = x$ and passes through (3, 2).

2. Find the area of the triangle with vertices at the point $(a,\ b+c), (b, c+a), (c, a+b)$.

3. Find the co-ordinates of foot of the perpendicular from the point (2, 4) on the line $x + y = 1$.

4. Let the area of a triangle is 5 and two of its vertices are A (2, 1) and B (3, –2). Then find third vertex of along which lies on the line $y = x + 3$.

5. Find the area of the quadrilateral whose sides are given by $|x| + |y| = 1$.

6. If the straight lines $x + y = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ form a triangle, then prove that it is isosceles.

7. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). Then find co-ordinates of the point A.

8. The equation to a pair of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. Then find equations to its diagonals.

9. The extremities of the diagonal of a square are (1, 1) and (–2, 1). Find the other two vertices and the equation of the other diagonal.

10. The equation to the base of an equilateral triangle ABC is $2x + y - 6 = 0$. The vertex is (3, 4). Find the equation of the other two sides and also the length of a side of the triangle.

11. Find the coordinates of the point which divides the line segment joining the points (2, 4) and (6, 8) in the ratio of 1 : 3 externally.

12. Find the area of a triangle ABC if the co-ordinates of the mid points of the sides of the triangle are (–1, –2), (6, 1) and (3, 5).

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $(-3, 5)$

PP2. $x + y - 2 = 0$

PP3. $9x + 8y = 72$

PP4. $\frac{7}{2}$

PP5. $\tan^{-1}\frac{15}{8}$

PP6. $\frac{9}{2}$

PP7. $x - 2y + 4\left(1 \pm \sqrt{5}\right) = 0$

PP8. $\frac{7}{5}$

PP9. $\frac{7}{10}$

PP10. $5x - 2y - 16 = 0$

ANSWERS TO EXERCISE – I

1. (–6, –5); (8, –3); (4, 1)

3. $3x + 4y = 9$

4. 3 and $\frac{11}{37}$

5. 5*x* J 12*y* + 53 = 0, 5*x* J 12*y* + 9 = 0

6. $\left(-\frac{1}{10}, \frac{37}{10}\right)$

7. $x\cos 60° + y\sin 60° = 2,\ p = 2$

8. (–3/2, 3/2) or (7/2, 13/2)

9. $\frac{1}{2}$

10. $\frac{4}{3}$

11. $\left(\frac{1}{2}, 3\right)$

12. –3

ANSWERS TO EXERCISE – II

1. $x + y = 5$

2. 0

3. $\left(-\frac{1}{2}, \frac{3}{2}\right)$

4. $\left(-\frac{3}{2}, \frac{3}{2}\right)$

5. 2 sq. units

7. $\left(\frac{13}{5}, 0\right)$

8. $4x + y = 13$ and $y = 4x - 7$

9. $\left(-\frac{1}{2}, \frac{5}{2}\right); \left(-\frac{1}{2}, -\frac{1}{2}\right); 2x + 1 = 0$

10. $m_1 = \frac{2+\sqrt{3}}{-1+2\sqrt{3}}$ and $m_2 = \frac{(\sqrt{3}-2)}{1+2\sqrt{3}}$, equation: y J 4 = m (x J 3); side = $\frac{8}{\sqrt{15}}$

11. (0, 2)

12. 74 sq. units