LESSON 9

SEQUENCES AND SERIES

Informally, a sequence is an ordered list of objects, but in this lesson the objects will usually be numbers.

e.g., 2, 0, –2, 2, 0, –2, 2, 0, –2, …………………;

1, 2, 4, 8, 16, ……………………..;

x, $x - 3$, $x - 6$, $x - 9$,………, *etc*.

You can observe that there is a pattern in each of these list of numbers/expressions. This pattern may be easy to observe and sometimes difficult. But with the help of this pattern we can tell the number at any position in the sequence.

It is obvious that there is no limit to the kinds of patterns we can form and thus infinite types of sequences can be generated. In this lesson, we will focus our attention to some basic sequences. Before that let us define certain terms.

Sequence : A set of numbers arranged in some definite order and according to some definite rule is called a sequence.

Alternatively, a real sequence is a function whose domain is the set of natural numbers *N* or a subset of *N,* and range is a set of real numbers.

Series : Terms of a sequence when added form a series.

Progression : When terms of a sequence are written under specific conditions, then the sequence is called a progression.

A progression is represented as t_1, t_2,……….,t_n,……….or a_1, a_2,……..,a_n,....... where t_1 (or a_1) means first term, and t_n (or a_n) means n^{th} term. t_k(or a_k) is called the general term of the progression. The number of terms can be finite or infinite.

$S_n = t_1 + t_2 + ... + t_n$, denotes the sum of corresponding series upto n terms.

Clearly, $t_n = S_n - S_{n-1}$.

1. ARITHMETIC PROGRESSION

A succession of terms is said to be in Arithmetic Progression if the difference between any term and its preceding term is independent of the position of the term.

Arithmetic Progression is abbreviated as A.P.

e.g., 5, 9, 13, 17, 21, ...;

$t, t - 5, t - 10, t - 15$,...........; *etc*.

We define,

Common difference = value of a term – value of the preceding term.

In the above examples, common difference (c.d.) is 4 and –5 respectively.

The first term is usually denoted by '*a*' and the common difference by '*d*'. So, an A.P. can be written as

$a, a + d, a + 2d, a + 3d,$ ....

If follows that the k^{th} term of an A.P., *i.e.,* t_k (or a_k) is given by

$t_k = a + (k - 1)d$ (General term)

If an A.P. consists of n terms, its n^{th} term is called its last term denoted by l and is given by

$l = t_n = a + (n - 1)d.$

Illustration 1

Question: **If p times the q^{th} term of an A.P. is equal to q times its p^{th} term, show that $t_{p-q} : t_{p+q} = p - q : p + q$.**

Solution: Let us proceed backwards.

We require $t_{p-q} : t_{p+q}$ in terms of p and q only.

If we assume the first term as '*a*' and common difference as '*d*', then we must obtain a relation between them to get the required ratio in that form.

Now, $\dfrac{t_{p-q}}{t_{p+q}} = \dfrac{a + (p - q - 1)d}{a + (p + q - 1)d}$

But from the given condition,

$pt_q = qt_p$

$\Rightarrow \quad p[a + (q - 1)d] = q[a + (p - 1)d]$

$\Rightarrow \quad (p - q)\, a = (pq - q - pq + p)d$

$\Rightarrow \quad a = d \qquad (\because p \neq q)$

Hence, $\dfrac{t_{p-q}}{t_{p+q}} = \dfrac{a + (p - q - 1)a}{a + (p + q - 1)a} = \dfrac{p - q}{p + q}$.

1.1 SUM OF AN ARITHMETIC PROGRESSION

If an A.P. contains n terms, the corresponding series can be summed.

Sum to n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}(a+l)$$

Proof :

Consider the following addition:

S = 2 + 8 + 14 + 20 + 26 + 32 + 38

S = 38 + 32 + 26 + 20 + 14 + 8 + 2

Adding $2S$ = (40 + 40 + 40 + 40 + 40 + 40 + 40)

i.e., $2S = 7 \times 40$

$$S = \frac{7}{2}\times 40 = \frac{7}{2} \text{ (first term + last term)}$$

$$= \frac{n}{2} \text{ (first term + last term)}$$

where, n is the number of terms of the A.P.

last term = t_n = a + (n J 1)d

$$\therefore \quad S = \frac{n}{2}[a+t_n] = \frac{n}{2}[a+a+(n-1)d] = \frac{n}{2}[2a+(n-1)d]$$

The above formula gives us the sum of n terms of an A.P.

Illustration 2

Question: **The sum of the first 13 terms of an A.P. is 21 and the sum of first 21 terms is 13. Find the sum of first 34 terms.**

Solution: Given $S_{13} = 21$ and $S_{21} = 13$

i.e., $\frac{13}{2}[2a+(13-1)d] = 21$

$\Rightarrow \quad a+6d = \frac{21}{13}$ ...(i)

and $\frac{21}{2}[a+(21-1)d] = 13$

$\Rightarrow \quad a+10d = \frac{13}{21}$ ...(ii)

From equation (i) & (ii),

$a = \frac{849}{273}$ and $d = -\frac{68}{273}$

$\therefore S_{34} = \frac{34}{2}[2a+(34-1)d] = -34$

Illustration 3

Question: **Find the arithmetic progression, consisting of 10 terms if the sum of the terms occupying the even places is equal to 15 and the sum of those occupying the odd places is equal to $12\frac{1}{2}$.**

Solution: Let the successive terms of the A.P. be $a_1, a_2, \ldots, a_9, a_{10}$.

By hypothesis $a_2 + a_4 + a_6 + a_8 + a_{10} = 15$

i.e., $(a + d) + (a + 3d) + \ldots + (a + 9d) = 15$

i.e., $5a + 25d = 15$ … (i)

and $a_1 + a_3 + a_5 + a_7 + a_9 = 12\frac{1}{2}$

$5a + 20d = 12\frac{1}{2}$ … (ii)

From (i) and (ii), we get $5d = 2\frac{1}{2}$ or $d = \frac{1}{2}$ and $a = \frac{1}{2}$

Hence the A.P. is $\frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$.

1.2 IMPORTANT FACTS

(i) If a constant is added to (or subtracted from) every term of an A.P., the resulting sequence will also be an A.P. with the same common difference;

(ii) If every term of an A.P. is multiplied by (or divided by) a fixed constant, the resulting sequence will also be an A.P.

(iii) It is convenient in problems dealing with only three terms to take the three terms in A.P. as a J*d, a, a + d*, with a as the middle term and d as the common difference.

(iv) Four terms in A.P. can be taken as a – 3*d*, *a* J *d*, *a + d, a* + 3*d*. Note that 2*d* is the common difference.

Illustration 4

Question: **Find four numbers in A.P. such that their sum is *50* and greatest of them is *4* times the least.**

Solution : Let the four numbers in *A.P.* be ~ J *3* , ~ J , ~ + , ~ + *3* .

Given, ~ J *3* + ~ J + ~ + + ~ + *3* = *50*

or *4~ = 50* $\therefore \alpha = \frac{25}{2}$ and ~ + *3* = *4(~ J 3)* or *3~ = 15*

or $\beta = \frac{\alpha}{5} = \frac{25}{5 \times 2} = \frac{5}{2}$

Hence, the four numbers are 5, 10, 15, 20.

Illustration 5

Question: **If the numbers $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ form an A.P., then prove that a^2, b^2, c^2 are also in A.P.**

Solution: $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow (c + a)(a + b), (b + c)(a + b), (b + c)(c + a)$ are in A.P.

[by multiplying by $(b + c)(c + a)(a + b)$]

$\Rightarrow (a^2 + ab + bc + ca), (b^2 + ba + bc + ac), (c^2 + ca + cb + ab)$ are in A.P.

$\Rightarrow a^2, b^2, c^2$ are in A.P. (Subtracting ab + bc + ca from all the three terms)

2. GEOMETRIC PROGRESSION

A succession of terms is said to be in Geometric Progression if the ratio of a term to its preceding term is independent of the position of the term. Geometric Progression is abbreviated as G.P.

e.g., 3, 15, 75, 375,............;

$t, t/3, t/9, t/27$, ...; *etc.*

We define

$$\text{Common ratio} = \frac{\text{value of a term}}{\text{value of its preceding term}}$$

In the above examples, common ratio (c.r.) is 5 and 1/3 respectively.

The first term is usually denoted by '*a*' and the common ratio by '*r*'. So, *a* G.P. can be written as

a, ar, ar^2, ar^3, ... $(r \neq 0)$

No term of a G.P. must be zero.

So, k^{th} term of *a* G.P. is given by

$t_k = ar^{k-1}$ (General term)

For a G.P. consisting of *n* terms, the last term is

$l = t_n = ar^{n-1}$

Illustration 6

Question: **If the fifth term of a G.P. is 81 and second term is 24, find the G.P.**

Solution: Here $t_5 = 81$ and $t_2 = 24$

$\therefore t_n = ar^{n-1}$ $\therefore$ $81 = ar^4$ ...(i)

and $24 = ar^{2-1}$ or $24 = ar$ ...(ii)

dividing (i) by (ii), we get

$$\frac{ar^4}{ar} = \frac{81}{24} = \frac{27}{8} \quad \text{or} \quad r^3 = \left(\frac{3}{2}\right)^3 \quad \therefore \quad r = \frac{3}{2}$$

Putting the value of *r* in (ii), we get $a = \dfrac{24}{r} = 24.\dfrac{2}{3} = 16$

Hence the required G.P. is 16, 24, 36, 54,...... .

2.1 SUM OF TERMS IN G.P.

Sum to n terms of a G.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ provided } r \neq 1$$

The sum formula shows that if the magnitude of r is less than 1, r^n decreases as n increases. If n becomes very large, *i.e.,* as $n \to \infty,\ r^n \to 0.$ So, if the number of terms is very large, we can get the sum of a G.P. to infinite terms, as a finite quantity.

$$\therefore \quad S_\infty = \frac{a}{1-r} \qquad \text{if} \qquad 0 < |r| < 1$$

Proof :

Let $S_n = a + ar + ar^2 + \ldots\ldots\ldots\ldots + ar^{n\ J\ 1}$ ...(i)

Consider two cases : $r = 1$, $r \div 1$

If $r = 1$, $S_n = \underbrace{a + a + a + \ldots\ldots\ldots + a}_{n \text{ times}} = na$

If $r \div 1$, multiplying (i) by r, we have

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^n$ ...(ii)

Subtracting (ii) from (i), we get

$$(1-r)S_n = a - ar^n \qquad \Rightarrow \qquad S_n = \frac{a(1-r^n)}{1-r}$$

For convenience, we write $S_n = \dfrac{a(r^n - 1)}{r-1}$ if $|r| > 1$.

$$\therefore \qquad S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r} & \textit{if} \quad -1 \le r < 1,\ r \neq 0 \\ na & \textit{if} \quad r = 1 \\ \dfrac{a(r^n - 1)}{r-1} & \textit{if} \quad |r| > 1 \textit{ i.e., } r < -1 \textit{ or } r > 1 \end{cases}$$

What happens if r = J1 ?

In this case the G.P. becomes a, Ja, a, $-a$, ...

So the sum depends on the number of terms.

$S_n = 0$ if n = 2, 4, 6, ...*i.e.,* when n is even

and $\quad S_n = a$ if n = 1, 3, 5, ...*i.e.,* when n is odd.

The formula $S_n = \dfrac{a(1-r^n)}{1-r}$ holds in this case.

Let us apply the S_n formula for the series with

$a = 10$ and $r = -\dfrac{1}{2}$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{(10)\left[1-\left(-\frac{1}{2}\right)^n\right]}{1-\left(-\frac{1}{2}\right)}$$

The quantity $\left(-\frac{1}{2}\right)^n$ becomes nearly zero if n is very large.

$$\therefore\ S_\infty = \frac{10(1-0)}{1+\frac{1}{2}} = \frac{20}{3}$$

So, the sum of infinite number of terms is finite.

Therefore, *a* G.P. can be summed till infinity if | *r* | < |.

(because, as $n \to \infty,\ r^n \to 0$)

Hence, $S_\infty = \frac{a}{1-r}$, if $|r| < 1$ *i.e.*, $-1 < r < 1$.

Illustration 7

Question: **Find the least value of *n* for which $1 + 3 + 3^2 + ...+ 3^{nJ1} > 1000$.**

Solution : Given, $1 + 3 + 3^2 + ...+ 3^{nJ1} > 1000$

or $1\left(\frac{3^n - 1}{3 - 1}\right) > 1000$

or 3^n J $1 > 2000$ or $3^n > 2001$ ...(i)

Now $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$, clearly $2187 > 2001$,

$\therefore$ least value of n for which $3^n > 2001$ is 7.

Illustration 8

Question: **If *s* is the sum to infinity of a G.P. whose first term is a, then prove that the sum of the first *n* terms is $\left[1-\left(1-\frac{a}{s}\right)^n\right]s$.**

Solution: If r be the common ratio, we have $s = \frac{a}{1-r}$; $\therefore r = 1 - \frac{a}{s}$

$$S_n = a\left[\frac{1-r^n}{1-r}\right] = a\left[\frac{1-\left(1-\frac{a}{s}\right)^n}{1-\left(1-\frac{a}{s}\right)}\right] = \frac{a\left[1-\left(1-\frac{a}{s}\right)^n\right]}{\frac{a}{s}}$$

$$S_n = \left[1-\left(1-\frac{a}{s}\right)^n\right]s$$

2.2 SOME RESULTS FOR G.P.

1. If a, b, c are in G.P., then for k (÷ 0)

   ak, bk, ck are also in G.P. $a/k, b/k, c/k$ are also in G.P.

2. If $a_1, a_2, a_3,$ …………….. and b_1, b_2, b_3 ………………… are two G.P.'s, then the sequence $a_1b_1, a_2b_2, a_3b_3;$ ……………..is also in G.P.

3. Three terms in a G.P. are taken as $\frac{a}{r}, a, ar.$

4. Four terms in a G.P., are taken as $\frac{a}{r^3}, \frac{a}{r}, a, ar^3.$

5. If a, b, c and d are in G.P., then $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} \Rightarrow b^2 = ac$, $c^2 = bd, ad = bc.$

6. Given a_1, a_2, a_3, …….$(a_i > 0\ \forall i)$ are in G.P. then $\log a_1$, $\log a_2$, $\log a_3$,......... are in A.P. and vice versa.

Illustration 9

Question: **If the product of three number in G.P. be 216 and their sum is 19, find the numbers.**

Solution : Let the three numbers be $\frac{a}{r}, a, ar$

Given, $\frac{a}{r}.a.ar = 216$ or $a^3 = 6^3$ $\therefore a = 6$

Also $\frac{a}{r} + a + ar = 19$ $\therefore \frac{6}{r} + 6 + 6r = 19$

or $6r^2\ J 13r + 6 = 0$ or $6r^2 - 9r - 4r + 6 = 0$

or $3r(2r\ J\ 3)\ J\ 2(2r\ J\ 3) = 0$ or $(3r\ J\ 2)(2r\ J\ 3) = 0$

$\therefore r = \frac{2}{3}, \frac{3}{2}$

When $r = \frac{2}{3}$, numbers will be 9, 6 and 4

when $r = \frac{3}{2}$, numbers will be 4, 6, 9. Hence the numbers are 4, 6, 9.

3. ARITHMETIC MEANS

If we are given two numbers, say a and b, then the number A which makes the progression a, A, b an arithmetic progression is called the Arithmetic mean of a and b.

Clearly, $A\ J\ a = b\ J\ A$

$$\Rightarrow \quad A = \frac{a+b}{2}$$

Here we have inserted only one number between a and b. We can also insert n numbers (say A_1, A_2,....,A_n) between a and b such that the progression a, A_1, A_2,......, A_n, b becomes arithmetic. There values can be obtained in terms of a and b as follows:

Consider the progression

$a, A_1, A_2, \ldots, A_n, b$

Here, first term = a, last term = b and number of terms = n + 2

Let d = common difference of A.P.

$\therefore \quad b = a + (n + 2 - 1)d$

$$\Rightarrow \quad d = \frac{b-a}{n+1}$$

So, $A_1 = a + d \quad = a + \frac{b-a}{n+1}$

$A_2 = a + 2d \quad = a + 2\frac{b-a}{n+1}$

$\vdots$

$A_n = a + nd \quad = a + n\frac{b-a}{n+1}$

So, the k^{th} arithmetic mean is given by

$$A_k = a + \frac{k(b-a)}{n+1} \qquad \text{where } k = 1, 2, \ldots, n$$

e.g., the 5^{th} of the 7 arithmetic means between the numbers 18 and 90 is given by

$$A_5 = 18 + \frac{5(90-18)}{7+1} = 63$$

If we have n numbers, a_1, a_2,......., a_n then arithmetic mean of these numbers is defined as

$$A = \frac{a_1 + a_2 + \ldots\ldots + a_n}{n}.$$

Illustration 10

Question: **If n arithmetic means are inserted between 20 and 80 such that first mean: last mean = 1 : 3, find n.**

Solution: Given $A_1 : A_n = 1 : 3$

$$\Rightarrow \quad \frac{20 + \frac{(1)(80-20)}{n+1}}{20 + \frac{(n)(80-20)}{n+1}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{20(n+1)+60}{20(n+1)+60n} = \frac{1}{3} \quad \Rightarrow \quad \frac{n+4}{4n+1} = \frac{1}{3} \quad \Rightarrow \quad n = 11.$$

4. GEOMETRIC MEANS

The geometric mean (G.M.) between two non-zero numbers a and b such that $ab > 0$ is the number G such that a, G, b are in G.P.

Hence $\frac{G}{a} = \frac{b}{G}$ or $G^2 = ab$ or $G = \sqrt{ab}$

The geometric mean of n positive numbers $a_1, a_2, \ldots, a_n$ is given by $G = \sqrt[n]{a_1 a_2 \ldots a_n}$

If $G_1, G_2, G_3, \ldots, G_n$ be n geometric means (G.M.s) between a and b, then $a, G_1, G_2, \ldots, G_n, b$ are in G.P.

If r be the common ratio of this sequence, then $b = t_{n+2} = ar^{n+1}$

Hence $r^{n+1} = \frac{b}{a}$ and $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

The k^{th} geometric mean is given as $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$.

This enables us to find $G_1, G_2, \ldots, G_n$.

Illustration 11

Question: **Show that the product of n geometric means inserted between two positive quantities is equal to the n^{th} power of the single geometric mean inserted between them.**

***Solution* :** Let $x_1, x_2, \ldots\ldots, x_n$ be the n G.M.'s between two given positive quantities a and b, then

$$x_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, x_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, x_3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, x_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Now $x_1 . x_2 \ldots\ldots\ldots\ldots x_n = a^n . \left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}+\ldots\ldots\ldots\ldots+\frac{n}{n+1}}$

$$= a^n \left(\frac{b}{a}\right)^{\frac{1+2+\ldots.+n}{n+1}} = a^n . \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n . \left(\frac{b}{a}\right)^{\frac{n}{2}} = a^{n-\frac{n}{2}}(b)^{\frac{n}{2}} = (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

$= (G.M.$ of a and $b)^n$ [$\because$ G.M. of a and $b = \sqrt{ab}$]

5. RELATION BETWEEN A.M., G.M. AND H.M. BETWEEN TWO REAL AND UNEQUAL QUANTITIES

Let a and b be two real positive and unequal quantities and A, G and H be the single A.M., G.M. and H.M. respectively between them.

Then $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$

Now $AH = \frac{a+b}{2} . \frac{2ab}{a+b} = ab = G^2 \quad \therefore \frac{G}{A} = \frac{H}{G}$ ...(i)

Hence ***A, G, H* are in G.P.**

Again, A J G = $\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}.\sqrt{b}}{2}$

$= \frac{(\sqrt{a} - \sqrt{b})^2}{2} > 0 \qquad [\because a \neq b]$

Thus A J $G > 0$ or $A > G$ ...(ii)

From (i), $\frac{H}{G} = \frac{G}{A}$ ... (iii)

From (ii) , $\frac{G}{A} < 1 \quad \therefore \frac{H}{G} < 1 \Rightarrow H < G \Rightarrow G > H$ ...(iv)

from (i) and (iv), we get $A>G>H$

Thus A.M., G.M. and H.M. between two real positive and unequal quantities are in G.P. and ***A.M. > G.M > H.M.***

Note :

1. A J G = $\frac{1}{2}(\sqrt{a} - \sqrt{b})^2 = 0$ y $A = G$ ę $a = b$

Hence $A = G = H$ ę $a = b$

2. The relation $A \geq G \geq H$ holds for n positive real number also.

Illustration 12

Question: **If one geometric mean *G* and two arithmetic means p and q are inserted between the quantities a and b, show that $G^2 = (2p$ J $q)(2q$ J $p)$.**

Solution : $G = \sqrt{ab}$, $p = \frac{2a+b}{3}$, $q = \frac{a+2b}{3}$

Now $(2p$ J $q)(2q$ J $p)$ = $\left(\frac{4a+2b-a-2b}{3}\right)\left(\frac{4b+2a-b-2a}{3}\right) = ab = G^2$

Illustration 13

Question: **If $x > 0$, prove that $x + \frac{1}{x} \geq 2$.**

Solution: $\because$ A.M. $\geq$ G.M.

$$\therefore \quad \frac{x + \frac{1}{x}}{2} \geq \sqrt{x.\frac{1}{x}} \Rightarrow \frac{x + \frac{1}{x}}{2} \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

Note : The least value of the sum of a positive number and its reciprocal is *2*.

6. SUMMATION OF SERIES

6.1 SUM OF THE FIRST N NATURAL NUMBERS

$1 + 2 + 3 + \ldots\ldots + n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$. This is evident as the numbers are in A.P.

6.2 SUM OF THE SQUARES OF THE FIRST N NATURAL NUMBERS

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: Consider the identity $(x + 1)^3 - x^3 = 3x^2 + 3x + 1$

Putting $x = 1, 2, 3, \ldots\ldots, n$, we have

$(n + 1)^3 - n^3 = 3n^2 + 3n + 1$

$n^3 - (n - 1)^3 = 3(n - 1)^2 + 3(n - 1) + 1$

…………………………………

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$

Adding $(n+ 1)^3 - 1^3 = 3\sum_{1}^{n} r^2 + 3\sum_{1}^{n} r + n$

Hence $3\sum_{1}^{n} r^2 = (n^3 + 3n^2 + 3n) - \frac{3n(n+1)}{2} - n$

$$\therefore \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

6.3 SUM OF THE CUBES OF THE FIRST N NATURAL NUMBERS

$$1^3 + 2^3 + \ldots\ldots + n^3 = \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

This can be proved by using the identity $(x + 1)^4 - x^4 \equiv 4x^3 + 6x^2 + 4x + 1$

Note: $\Sigma n^3 = (\Sigma n)^2 = \dfrac{n^2(n+1)^2}{4}$

Certain series can be summed up if each term of the series can be expressed as the difference of two consecutive terms of another series known as ancillary series. This is illustrated in the solved examples ahead.

Illustration 14

Question: **Find the sum of n terms of the series whose nth term is $12n^2 - 6n + 5$.**

Solution: Let $t_n = 12n^2 - 6n + 5$

$$\therefore \quad S_n = \sum_{n=1}^{n} t_n = \sum_{n=1}^{n}\left(12n^2 - 6n + 5\right) = \sum_{n=1}^{n} 12n^2 - \sum_{n=1}^{n} 6n + \sum_{n=1}^{n} 5$$

$$= 12\sum_{n=1}^{n} n^2 - 6\sum_{n=1}^{n} n + 5n = 12\,\frac{n(n+1)(2n+1)}{6} - \frac{6n(n+1)}{2} + 5n$$

$$= 2n\,(2n^2 + 3n + 1) - 3n\,(n+1) + 5n = 4n^3 + 3n^2 + 4n$$

Illustration 15

Question: **Find the sum of the series $1.n + 2.(n - 1) + 3.(n - 2) + \ldots\ldots + n.1$.**

Solution: Here n occurs in each term, therefore, for general term we will find the r^{th} term

Since r^{th} term of the sequence 1, 2, 3, …… = $1 + (r - 1).1$

and r^{th} term of the sequence $n, n - 1, n - 2, \ldots\ldots = n + (r - 1)(-1)$

Now r^{th} term $t_r = \{1 + (r-1)\cdot 1\}\{n + (r-1)(-1)\}$

$$= r\,(n - r + 1) = nr - r^2 + r$$

$$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(nr - r^2 + r\right)$$

$$= n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\left(n - \frac{2n+1}{3} + 1\right)$$

$$= \frac{n(n+1)}{2}\cdot\frac{(3n - 2n - 1 + 3)}{3} = \frac{n(n+1)(n+2)}{6}$$

7. METHOD OF DIFFERENCES

Illustration 16

Question: **Find the n^{th} term and the sum to n terms of the series**

1 + 6 + 23 + 58 + 117 + 206+ ………

Solution:

The terms of the series	1	6	23	58	117	206	…
	u_1	u_2	u_3	u_4	u_5	u_1	…

The 1st order of differences	$(u_2 - u_1)$	$(u_3 - u_2)$	$(u_4 - u_3)$	$(u_5 - u_4)$	$(u_6 - u_5)$	…
	5	17	35	59	89	…
	v_2	v_2	v_3	v_4	v_5	
The 2nd order of differences	$(v_2 - v_1)$	$(v_3 - v_2)$	$(v_4 - v_3)$	$(v_5 - v_4)$		
	12	18	24	30		
The 3rd order of differences	6	6	6			

It may be noted that the terms of the successive order of difference series are obtained from the immediately preceding series by taking the difference of two consecutive terms.

If at any stage, in finding the successive order of difference series terms, all the terms reduce to the same number (as in this problem, this happens at the 3rd order of differences), then the nth term u_n of the given series is a polynomial in n of degree equal to that order of difference series whose terms are all the same. Thus in this problem u_n is of degree 3 and hence u_n can be taken as either $an^3 + bn^2 + cn + d$ or conveniently as $u_n = An(n-1)(n-2) + Bn(n-1) + Cn + D$

The constants A, B, C, D are determined as follows

put $n = 1$, $u_1 = 1$ and $u_1 = C + D$ y $C + D = 1$

put $n = 2$, $u_2 = 6$ and $u_2 = 2B + 2C + D$ $\therefore 2B + 2C + D = 6$

put $n = 3$, $u_3 = 23$ and $u_3 = 6A + 6B + 3C + D$ $\therefore 6A + 6B + 3C + D = 23$

put $n = 4$, $u_4 = 58$ and $u_4 = 24A + 12B + 4C + D$ $\therefore 24A + 12B + 4C + D = 58$

Solving for A, B, C, D, we get $A = 1, B = 3, C = -1, D = 2$

Hence $u_n = n(n-1)(n-2) + 3n(n-1) - n + 2$

$$= n^3 - 2n + 2$$

$$\therefore \sum_{n=1}^{n} u_n = \Sigma n^3 - 2\Sigma n + 2\Sigma 1$$

$$= \frac{n^2(n+1)^2}{4} - \frac{2n(n+1)}{2} + 2n$$

$$= \frac{n}{4}\{n^3 + 2n^2 - 3n + 4\}$$

Illustration 17

Question: **Sum to *n* terms the series 1 + 4 + 11 + 26 + 57 + 120 + ………………**

Solution:

	1	4	11	26	57	120
First order of differences	3	7	15	31	63	
Second order of differences	4	8	16	32		

The 2nd order of difference is a G.P. of common ratio 2.

In this case, $u_n = A\,(\text{common ratio})^n$ + (first degree polynomial in n since the 2nd order difference is a G.P.)

i.e., $u_n = A.2^{n-1} + Bn + C$

obtain A, B and C using u_1, u_2 and u_3.

then

$$S_n = \sum_{n=1}^{n} u_n = A(2^\circ + 2^1 + 2^2 + ...2^{n-1}) + B\sum n + C\sum(1) = A(2^n - 1) + \frac{Bn(n+1)}{2} + Cn$$

PRACTICE PROBLEMS

PP1. Find the 15th term of the A.P. $x - 7$, x J 2, $x + 3$, ……….

PP2. Find the sum to 15 terms of the A.P. x J 2, $x + 3$, $x + 8$, …… at $x = 10$.

PP3. Three numbers in A.P. have sum 24. Then find the middle term.

PP4. Find the common ratio of a G.P. having 10th term and 1st term equal to 1536 and J3.

PP5. Between 1 and 31, m arithmetic means have been inserted in such a way that the ratio of the 7th and $(m-1)$th means is 5 : 9. Find the value of m.

PP6. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

PP7. Find 5 A.M. which are inserted between 10 and 100.

PP8. Find 4 G.M.s which are inserted between 5 and 160.

PP9. Find the minimum value of $\tan\theta + \cot\theta$, $0 < \theta < \frac{\pi}{2}$.

PP10. The A.M. of two numbers exceeds their G.M. by 2 and the G.M. exceeds their H.M. by $\frac{8}{5}$, find the numbers.

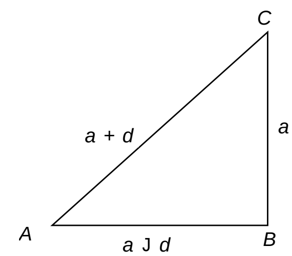
SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

If the sides of a right triangle are in A.P., then find the sum of the sines of the two acute angles.

Solution :

Let the sides be a J d, a $a + d$ as shown ; $d > 0$,



then, $(a + d)^2 = a^2 + (a$ J $d)^2$

$\Rightarrow \quad a^2 = 4ad \qquad \Rightarrow \quad a = 4d.$

Hence, the three sides are $4d$ J d, $4d$, $4d + d$ i.e. $3d, 4d, 5d$.

$$\therefore \quad \sin A + \sin C = \frac{a}{a+d} + \frac{a-d}{a+d} = \frac{4d}{5d} + \frac{3d}{5d} = \frac{7}{5}.$$

***Example* 2:**

Find the sum of *n* terms of the series $\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \ldots\ldots$

Solution :

Required sum $= \frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \ldots\ldots + \frac{1}{\sqrt{3n-1}+\sqrt{3n+2}}$

($\because$ n^{th} term of the sequence 2, 5, 8, …… is $3n - 1$ and that of 5, 8, 11, …..*is* $3n + 2$)

$$= \frac{\sqrt{5}-\sqrt{2}}{5-2} + \frac{\sqrt{8}-\sqrt{5}}{8-5} + \frac{\sqrt{11}-\sqrt{8}}{11-8} + \ldots\ldots + \frac{\sqrt{3n+2}-\sqrt{3n-2}}{3n+2-(3n-1)}$$

$$= \frac{1}{3}\left\{\sqrt{5}-\sqrt{2}+\sqrt{8}-\sqrt{5}+\sqrt{11}-\sqrt{8}+\ldots\ldots+\sqrt{3n+2}-\sqrt{3n-1}\right\}$$

$$= \frac{1}{3}\left[\sqrt{3n+2}-\sqrt{2}\right].$$

***Example* 3:**

If $|x| < 1$, then find $1 + 3x + 5x^2 + 7x^3 + \ldots$ to ã .

Solution :

Let

$S = 1 + 3x + 5x^2 + 7x^3 + \ldots..$ ...(i)

then x $S = 1x + 3x^2 + 5x^3 + \ldots\ldots$ ...(ii)

Subtracting (ii) from (i) , we get

S J $xs = 1 + 2x + 2x^2 + 2x^3 + \ldots\ldots$ to $\infty \Rightarrow S\,(1$ J $x) = 1 + (2x + 2x^2 + 2x^3 + \ldots..$ to ∞

$\Rightarrow \quad S(1$ J $x) = 1 + \frac{2x}{1-x} = \frac{1-x+2x}{1-x} \Rightarrow S = \frac{1+x}{(1-x)^2}$

***Example* 4:**

Find the eleventh term of the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, …..

Solution:

Each term, beginning with 3rd, is the sum of two previous terms *i.e.,*

$T_3 = T_1 + T_2$, $T_4 = T_2 + T_3$,...... *etc*. $\therefore$ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.............

***Example* 5:**

The sum of four numbers in G.P. is 60 and the arithmetic mean of the first and last numbers is 18. Find the numbers.

Solution:

Let the four numbers be a, ar, ar^2, ar^3

$$a(1 + r + r^2 + r^3) = 60 \qquad \dots (i)$$

$$\frac{a + ar^3}{2} = 18 \qquad \dots (ii)$$

$\therefore a(1 + r)(1 + r^2) = 60$ and $a(1 + r)(1 - r + r^2) = 36$

Division gives $\frac{1 + r^2}{1 - r + r^2} = \frac{5}{3}$

$\therefore 2r^2 - 5r + 2 = 0$ giving $r = 2$ or $\frac{1}{2}$ Putting $r = 2$ in equation (i), we get $a = 4$

$\therefore$ the numbers are 4, 8, 16, 32.

$r = \frac{1}{2}$ gives the same four numbers.

***Example* 6:**

Sum to n terms the series

$$\mathbf{(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \dots}$$

Solution:

Let $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^2) + \dots$ upto n terms

Observe that multiplying both sides by (x – y), the terms on R. H. S. get reduced to

$x^2 - y^2, x^3 - y^3, x^4 - y^4, \dots\dots\dots$

$\therefore (x \text{ J } y)(S_n) = (x^2 \text{ J } y^2) + (x^3 - y^3) + (x^4 - y^4) + \dots + (x^{n+1} - y^{n+1})$

$$= (x^2 + x^3 + \dots + x^{n+1}) - (y^2 + y^3 + \dots + y^{n+1})$$

$$= x^2\frac{(1 - x^n)}{1 - x} - y^2\frac{(1 - y^n)}{1 - y}$$

$$\therefore S_n = \frac{1}{x - y}\left[\frac{x^2(1 - x^n)}{1 - x} - y^2\frac{(1 - y^n)}{1 - y}\right]$$

***Example* 7:**

Sum to *n* terms the series $\frac{1}{3\cdot7\cdot11}+\frac{1}{7\cdot11\cdot15}+....$

Solution:

Let $u_n = \frac{1}{(4n-1)(4n+3)(4n+7)}$

Consider $v_n = \frac{1}{(4n+3)(4n+7)}$ (Obtained by deleting the first factor $(4n-1)$ in the denominator of u_n.)

$v_{n-1} = \frac{1}{(4n-1)(4n+3)}$

Now $v_n - v_{n-1} = \frac{(4n-1)-(4n+7)}{(4n-1)(4n+3)(4n+7)} = -8u_n$

$\therefore u_n = \frac{-1}{8}(v_n - v_{n-1})$

$v_{n-1} = -\frac{1}{8}(v_{n-1} - v_{n-2})$

……………………………

$u_1 = -\frac{1}{8}(v_1 - v_0)$

Adding vertically $u_1 + u_2 + \ldots\ldots + u_n = -\frac{1}{8}(v_n - v_0) = \frac{1}{8}(v_0 - v_n)$

$$S_n = \frac{1}{8}\left(\frac{1}{21} - \frac{1}{(4n+3)(4n+7)}\right)$$

Note: In this case sum to infinity is also possible since the second term in $S_n \to 0$ as $n \to \infty$

$\therefore S_\infty = \frac{1}{168}$

***Example* 8:**

Sum to *n* terms

(i) 6 + 66 + 666 + ……. **(ii) .6 + .66 + .666 + ……**

Solution:

(i) Let $S = 6 + 66 + 666 + \ldots$..

$\frac{9S}{6} = 9 + 99 + 999 + \ldots\ldots. \quad = (10 - 1) + (10^2 - 1) + \ldots.. + (10^n - 1)$

$\frac{3S}{2} = (10 + 10^2 + \ldots. + 10^n)$ J $(1 + 1 + \ldots.\ n \text{ terms}) = \frac{10(10^n - 1)}{9} - n$

$S = \frac{2}{27}\left\{10^{n+1} - 9n - 10\right\}$

(ii) Let $S = .6 + .66 + .666 + \ldots\ldots$

$\frac{9S}{6} = .9 + .99 + .999 + \ldots\ldots.$

$= (1 - 10^{-1}) + (1 - 10^{-2}) + \ldots\ldots + (1 - 10^{-n}) = n - 10^{-1}\ \frac{\left(-10^{-n} + 1\right)}{-10^{-1} + 1} = n + \frac{10^{-n} - 1}{9}$

$\therefore \quad S = \frac{2}{27}\{9n - 1 + 10^{-n}\}$

***Example* 9:**

If the $(p + 1)^{th}$, $(q + 1)^{th}$, and $(r + 1)^{th}$ terms of an A.P. are in G.P; and p, q, r are in H.P., show that the ratio of the first term to the common difference of the A.P., is $-\frac{q}{2}$.

Solution:

If $a, a + d, \ldots.$ be the A.P., then the $(p + 1)^{th}$, $(q + 1)^{th}$, $(r + 1)^{th}$ terms are $a + pd, a + qd, a + rd$ and these are in G.P.

$\therefore \quad (a + qd)^2 = (a + pd)(a + rd)$

i.e., $2aqd + d^2p^2 = (p + r)ad + prd^2$

$(d \div 0)$and hence $d(q^2$ J $pr) = a\{p + r$ J $2q\}$ ...(i)

Using this in (i), $\frac{a}{d} = \frac{q^2 - pr}{\frac{2pr}{q} - 2q} = \frac{q}{2} \cdot \frac{q^2 - pr}{pr - q^2} = -\frac{q}{2}$

This is the desired result.

Example **10**

(a) If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$............ to ã = $\frac{®^2}{6}$, then obtain the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+$..... to ã.

(b) Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be an A.P. Find the value of the common difference of this A.P. that would make $a_2\, a_5\, a_8$ greatest.

Solution:

(a) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots \text{ to } \infty = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\text{to } \infty\right)$

$$-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\ldots\text{to } \infty\right) = \frac{\pi^2}{6}-\frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\text{to } \infty\right)$$

$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}.$$

(b) Let d be the common difference, then $a_2\, a_5\, a_8 = (a_1 + d)\,(a_1 + 4d)\,(a_1 + 7d)$

$= (a_1 + 4d$ J $3d)(a_1 + 4d)(a_1 + 4d + 3d)$

$= (a_5$ J $3d)(a_5)(a_5 + 3d)$

$= a_5(a_5^2$ J $9d^2)$

$= a_5(a_5^2$ J $(3d)^2)$

This product is greatest when $3d = 0$ *i.e.,* when $d = 0$

EXERCISE – I

1. (i) A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6,\ n \in N$. Show that the first three terms of the sequence are zero and all other terms are positive.

(ii) The *n*th term of a sequence is given by $a_n = 2n^2 + n + 1$. Show that it is not an A.P.

2. (i) Find the 18th term of the A.P. $\sqrt{2},\ 3\sqrt{2}, 5\sqrt{2},\ ......$

(ii) Which term of the A.P. 4, 9, 14, …… is 254?

(iii) How many terms are there in the A.P. 7, 10, 13, ……43?

3. (i) Find the 8th term of the G.P. 0.3, 0.06, 0.012, ……

(ii) Which term of the progression 18, –12, 8, …… is $\frac{512}{729}$?

4. Which term of the H.P. $4, \frac{30}{7}, \frac{60}{13},\ ......$ is 10.

5. (i) The 6th and 17th terms of an A.P. are 19 and 41 respectively, find the 40th term.

(ii) How many numbers of two digit are divisible by 3?

6. The first and the last terms of an A.P. are *a* and *l* respectively. Show that the sum of the *n*th term from the beginning and the *n*th term from the end is $a + l$.

7. Find the sum of the following arithmetic progression 50, 46, 42, …… to 10 terms.

8. Find the sum of the following arithmetic progression:

$\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y},\ ......$ to *n* terms

9. (i) Find the sum of all integers between 50 and 500 which are divisible by 7.

(ii) If 12th term of an A.P. is –13 and the sum of the first four terms is 24, what is the sum of first 10 terms?

10. (i) The product of three numbers in G.P. is 216. If 2, 8, 6 be added to them, the results are in A.P., find the numbers

(ii) The sum of three numbers *a, b, c* in A.P. is 18. If *a* and *b* are each increased by 4 and *c* is increased by 36, the new numbers form a G.P. Find *a, b, c*.

EXERCISE – II

1. (a) Find the sum of the following geometric progressions:

(i) 2, 6, 18, …… to 7 terms

(ii) $(a^2 - b^2), (a-b), \left(\frac{a-b}{a+b}\right),$ …… to n terms

(b) Evaluate the following:

(i) $\sum_{k=1}^{n} (2^k + 3^{k-1})$ (ii) $\sum_{n=2}^{10} 4^n$

2. If S_1 be the sum of $(2n+1)$ terms of an A.P. and S_2 be the sum of its odd terms, then prove that $S_1 : S_2 = (2n+1) : (n+1)$

3. (i) If $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

(ii) If *a, b, c* are in A.P., prove that $a^2 + c^2 + 4ac = 2(ab + bc + ca)$.

4. A.M.s are inserted between 7 and 71 in such a way that the 5th A.M. is 27. Find the number of A.M.s.

5. (i) Find the sum of the series 7 + 77 + 777 + …… to n terms.

(ii) Express the recurring decimal 0.125125125…… as a rational number.

6. (i) The ratio of the sum of first three terms is to that of first 6 terms of a G.P. is 125 : 152. Find the common ratio.

(ii) If mth term of a H.P. is n and nth term is m, then show that its $(m+n)$th term is $\frac{mn}{m+n}$.

(iii) Insert 6 harmonic means between 3 and $\frac{6}{23}$.

7. (i) Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.

(ii) If one A.M., A and two geometric means G_1 and G_2 inserted between any two positive numbers, show that $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A$.

(iii) If f is a function satisfying $f(x+y)=f(x)f(y)$ for all $x, y \in N$ such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$, find the value of n.

8. (i) Find the sum to *n* terms of the series $1+2x+3x^2+4x^3+......$

(ii) Find the sum to infinity of the series $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+...\infty$.

(iii) If *a, b, c*, *d* are in G.P., prove that $(a^n+b^n), (b^n+c^n), (c^n+d^n)$ are in G.P.

9. (i) Find the sum of the series to *n* terms $1.2^2+2.3^2+3.4^2+......$

(ii) Sum the series to *n* term $1+3+6+10+15+......$

(iii) 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

10. (i) Find the sum to *n* terms of the series 5 + 11 + 19 + 29 + 41 + …..

(ii) Find the sum to *n* terms of the series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+......$

(iii) Find the sum of the following series up to *n* terms $\frac{1^3}{1}+\frac{1^3+2^2}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+......$

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $x + 63$

PP2. 645

PP3. 8

PP4. J2

PP5. 14

PP6. $n = -\frac{1}{2}$

PP7. 25, 40, 55, 70, 85

PP8. 10, 20, 40, 80

PP9. 2

PP10. 16, 4

ANSWERS TO EXERCISE – I

2. (i) $35\sqrt{2}$

(ii) 51

(iii) 13

3. (i) $(0.3)(0.2)^7$

(ii) 9

4 10^{th} term

5. (i) 87

(ii) 30

7. 320

8. $\frac{n}{2(x+y)}[n(2x-y)-y]$

9. (i) 17696

(ii) 0

10. (i) 18, 6, 2 or 2, 6, 18

(ii) $a=-2, b=6, c=14$ or $a=46, b=6, c=-34$

ANSWERS TO EXERCISE – II

1. (a) (i) 2186

(ii) $\frac{a-b}{(a+b)^{n-2}}\left[\frac{(a+b)^n-1}{(a+b)-1}\right]$

(b) (i) $\frac{1}{2}\left(2^{n+2}+3^n-5\right)$

(ii) $\frac{16}{3}\left(4^9-1\right)$

4. 15

5. (i) $\frac{7}{81}[10^{n+1}-9n-10]$

(ii) $\frac{125}{999}$

6. (i) $\frac{3}{5}$

(iii) $\frac{6}{5},\frac{3}{4},\frac{6}{11},\frac{3}{7},\frac{6}{17},\frac{3}{10}$

7. (i) $\frac{16}{3}$, 8, 12, 18, 27

(iii) 4

8. (i) $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{(1-x)}$

(ii) 1

9. (i) $\frac{n}{12}(n+1)(n+2)(3n+5)$ (ii) $\frac{n}{6}(n+1)(n+2)$ (iii) 25 days

10. (i) $\frac{n}{3}(n+2)(n+4)$ (ii) $\frac{n}{2n+1}$ (iii) $\frac{n}{24}\left(2n^2+9n+13\right)$