LESSON 8

BINOMIAL THEOREM

1. BINOMIAL EXPRESSION

Any algebraic expression consisting of only two terms is known as binomial expression. The terms may consist of variables *x*, *y* etc. or constants or their mixed combinations. For example: 2*x* + 3*y*, 4*xy* + 5 etc.

2. BINOMIAL THEOREM FOR POSITIVE INDEX

Let us have a look at the following identities done earlier:

$(a+b)^0 = 1, \quad a+b \neq 0$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a+b)^2$, number of terms is 3 whereas the index of $(a+b)^2$ is 2.

(ii) Powers of the first quantity '*a*' go on decreasing by 1 whereas the powers of the second quantity '*b*' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of *a* and *b* is the same and is equal to the index of $a+b$.

We now arrange the coefficients in these expansions as follows:

Index	**Coefficients**
0	**1**
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**

Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our intersect.

We can extend the pattern given in figure writing a few more rows:

Index	Coefficients
0	**1**
1	**1** ď **1**
2	**1** ď **2** ď **1**
3	**1** ď **3** ď **3** ď **1**
4	**1** **4** **6** **4** **1**

Pascal's Triangle

The structure given in figure in looks like a triangle with 1 at the top vertex and running down to two slanting sides.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand $(2x+3y)^5$ by using Pascal's triangle. The row for index 5 is

1 5 10 10 5 1

Using this row and our observations (i), (ii) and (iii), we get

$$(2x+3y)^5 = (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

Now, if we want to find the expansion of $(2x+3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index.

For this, we make use of the concept of combinations studied earlier to rewirte the numbers in the Pascal's triangle. We know that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ and n is a non-negative integer. Also ${}^nC_0 = 1 = {}^nC_n$

The Pascal's triangle can now be rewritten as figure:

Index	**Coefficients**
0	${}^0C_0$ (= **1**)
1	${}^1C_0$ (= **1**) ${}^1C_1$ (= **1**)
2	${}^2C_0$ (= **1**) ${}^2C_1$ (= **2**) ${}^2C_2$ (= **1**)
3	${}^3C_0$ (= **1**) ${}^3C_1$ (= **3**) ${}^3C_2$ (= **3**) ${}^3C_3$ (= **1**)
4	${}^4C_0$ (= **1**) ${}^4C_1$ (= **4**) ${}^4C_2$ (= **6**) ${}^4C_3$ (= **4**) ${}^4C_4$ (= **1**)
5	${}^5C_0$ (= **1**) ${}^5C_1$ (= **5**) ${}^5C_2$ (= **10**) ${}^5C_3$ (= **10**) ${}^5C_4$ (= **5**) ${}^5C_5$ (= **1**)

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example:

For the index 7 the row would be

$${}^7C_0 \;\; {}^7C_1 \;\; {}^7C_2 \;\; {}^7C_3 \;\; {}^7C_4 \;\; {}^7C_5 \;\; {}^7C_6 \;\; {}^7C_7$$

Thus, using this row and the observations (i), (ii) and (iii), we have

$$(a+b)^7 = {}^7C_0 a^7 + {}^7C_1 a^6 b + {}^7C_2 a^5 b^2 + {}^7C_3 a^4 b^3 + {}^7C_4 a^3 b^4 + {}^7C_5 a^2 b^5 + {}^7C_6 ab^6 + {}^7C_2 b^7$$

An expansion of a binomial to any positive integral index say n can now be visualizes using these observations. We are now in a position to write the expansion of a binomial to any positive integral index.

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power.

In general for a positive integer n

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots.. + {}^nC_n x^0 y^n \text{ , where } {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

for $r = 0, 1, 2, \ldots\ldots , n$ is called binomial coefficient.

2.1 PROOF OF BINOMIAL THEOREM

The Binomial theorem can be proved by mathematical induction

Let $P(n)$ stands for the mathematical statement

$$(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + a^n \quad \ldots\text{(i)}$$

Note that there are $(n+1)$ terms in R.H.S. and all the terms are of the same degree in x and a together.

When $n = 1$, L.H.S. $= x + a$ and R.H.S. $= x + a$ (there are only 2 terms)

$\therefore$ P (1) is verified to be true

Assume $P(m)$ to be true

i.e., $(x+a)^m = x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a^2 + \ldots + {}^mC_r\, x^{m-1}\, a^r + \ldots + a^m$ …(ii)

Multiplying equation (ii) by $(x+a)$, we have

$$(x+a)^m\,(x+a) = (x+a)\left\{x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a^2 + \ldots + {}^mC_r\, x^{m-r}\, a^r + \ldots + a^m\right\}$$

i.e., $(x+a)^{m+1} = x^{m+1} + \left({}^mC_1 + 1\right)x^m a + \left({}^mC_2 + {}^mC_1\right)x^{m-1} a^2 + \ldots$

$$+ \left({}^mC_r + {}^mC_{r-1}\right)x^{m-r+1}a^r + \ldots + a^{m+1}$$

$= x^{m+1} + {}^{(m+1)}C_1\, x^m\, a + {}^{(m+1)}C_2\, x^{m-1}\, a^2 + \ldots$

$+ {}^{m+1}C_r\, x^{m+1-r}\, a^r + \ldots a^{m+1}$ …(iii)

(using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

Equation (iii) implies that $P(m+1)$ is true and hence by induction $P(n)$ is true.

Alternative method

By choosing x from all the brackets we get the term x^n. Choosing x from $(n-1)$ factors and 'a' from the remaining factor we get $x^{n-1}\, a$. But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e.,) ${}^nC_1$. Choosing x from $(n-2)$ factor and a from the remaining two factors, we get $x^{n-2}\, a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors. i.e., ${}^nC_2$. Finally choosing 'a' from all the factors we get the term a^n.

$$\therefore (x+a)^n = x^n + {}^nC_1 x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + a^n$$

Illustration 1

Question: **Expand** $\left(x - \frac{1}{x}\right)^6$

Solution:

$$\left(x - \frac{1}{x}\right)^6 = {}^6C_0 x^6 + {}^6C_1 x^5 \left(\frac{-1}{x}\right) + {}^6C_2 x^4 \left(\frac{-1}{x}\right)^2 + {}^6C_3 x^3 \left(\frac{-1}{x}\right)^3 + {}^6C_4 x^2 \left(\frac{-1}{x}\right)^4$$

$$+ {}^6C_5 x \left(\frac{-1}{x}\right)^5 + {}^6C_6 x^0 \left(\frac{-1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

3. GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term, given by $t_{r+1} = {}^nC_r\, x^{n-r}\, y^r$ where $r = 0, 1, 2 \ldots\ldots\ldots n$.

- Every term in the expansion is of nth degree in variables x and y.
- The total number of terms in the expansion is $n + 1$.
- Binomial expansion can also be expressed as $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$.

Illustration 2

Question: **Find the 11th term in the expansion of** $\left(3x - \frac{1}{x\sqrt{3}}\right)^{20}$.

Solution: The general term $= t_{r+1} = (-1)^r\ {}^{20}C_r\ (3x)^{20-r}\left(\frac{1}{x\sqrt{3}}\right)^r$

For the 11th term, we must take $r = 10$

$$\therefore t_{11} = t_{10+1} = (-1)^{10}\ {}^{20}C_{10}\ (3x)^{20-10}\left(\frac{1}{x\sqrt{3}}\right)^{10}$$

$$= {}^{20}C_{10}\ 3^{10}\ x^{10}\ \frac{1}{x^{10}\ (\sqrt{3})^{10}} = {}^{20}C_{10}\ \left(\sqrt{3}\right)^{10} = {}^{20}C_{10}\ 3^5$$

Illustration 3

Question: **Find the term independent of x in** $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

Solution: The general term $= {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(\frac{-1}{3x}\right)^r$

$$= (-1)^r\ {}^9C_r\ \frac{3^{9-2r}}{2^{9-r}} \cdot x^{18-3r}$$

The term independent of x, (or the constant term) corresponds to x^{18-3r} being x^0 or $18 - 3r = 0 \Rightarrow r = 6$

$\therefore$ the term independent of x is the 7^{th} term and its value is

$$(-1)^6\ {}^9C_6\ \frac{3^{9-12}}{2^{9-6}} = {}^9C_3\ \frac{3^{-3}}{2^3} = \frac{9.8.7}{3.2.1}\cdot\frac{1}{(6)^3} = \frac{7}{18}$$

4. MIDDLE TERMS OF THE EXPANSION

In the binomial expansion of $(x + y)^n$

4.1 WHEN *n* IS ODD

There are $(n + 1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \qquad \text{for} \qquad r = \frac{n-1}{2}$$

and $$t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}} y^{\frac{n+1}{2}} \qquad \text{for} \qquad r = \frac{n+1}{2}$$

Are the binomial coefficients of the two middle terms always numerically equal?

4.2 WHEN *n* IS EVEN

There are odd terms in the expansion and hence only one middle term is given by

$$t_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2} \qquad \text{for} \qquad r = \frac{n}{2}$$

Illustration 4

Question: **Find the middle term in the expression of (1 J 2*x* + *x*^{2})^n.**

Solution: $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$

Here $2n$ is even integer, therefore, $\left(\frac{2n}{2} + 1\right)^{\text{th}}$ i.e. $(n + 1)^{\text{th}}$ term will be the middle term.

Now $(n + 1)^{\text{th}}$ term in $(1 - x)^{2n} = {}^{2n}C_n\, (1)^{2n-n}\, (-x)^n$

$$= {}^{2n}C_n(-x)^n = \frac{(2n)!}{n!\,n!}(-x)^n.$$

5. BINOMIAL COEFFICIENTS

In the binomial expansion of $(x + y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$

- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

What is the greatest coefficient in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$?

Illustration 5

Question: **If the coefficient of $(2r + 4)^{th}$ term and (r J 2)th term in the expansion of $(1 + x)^{18}$ are equal, find r.**

Solution: Since coefficient of $(2r + 4)^{th}$ term in $(1 + x)^{18} = {}^{18}C_{2r+3}$.

Coefficient of $(r - 2)^{th}$ term $= {}^{18}C_{r-3}$

$\Rightarrow \quad {}^{18}C_{2r+3} = {}^{18}C_{r-3} \quad \Rightarrow \quad 2r + 3 + r - 3 = 18 \quad \Rightarrow \quad 3r = 18 \quad \Rightarrow \quad r = 6.$

- The ratio of $(r + 1)^{th}$ coefficient to r^{th} coefficient is $\frac{n-r+1}{r}$ as

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r+1)!\,(r-1)!}} = \frac{n-r+1}{r}$$

Illustration 6

Question: **Prove that the sum of the coefficients in the expansion of (1 + x J 3x^2)^{2163} is J1.**

Solution: Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients

$= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1.$

Illustration 7

Question: **If the sum of the coefficients in the expansion of (~x^2 J 2x + 1)^{35} is equal to the sum of the coefficients in the expansion of (x J ~y)^{35}, then find the value of ~.**

Solution: Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$

$\therefore \quad (\alpha - 1)^{35} = (1 - \alpha)^{35}$

$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35}$

$\Rightarrow \quad 2(\alpha - 1)^{35} = 0$

$\therefore \quad \alpha - 1 = 0$

$\therefore \quad \alpha = 1$

- **The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}**

Since $(1 + x)^n = C_0 + C_1 + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$

Putting $x = -1$,

$$0 = C_0 - C_1 + C_2 - C_3 + \ldots\ldots + (-1)^n C_n$$

and $2^n = C_0 + C_1 + C_2 + C_3 + \ldots\ldots + C_n$ {from (i)}

Adding and subtracting these two equations, we get

$$2^n = 2(C_0 + C_2 + C_4 + \ldots\ldots) \text{ and } 2^n = 2(C_1 + C_3 + C_5 + \ldots\ldots)$$

$\therefore \quad C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

sum of coefficients of odd terms = sum of coefficients of even terms = 2^{n-1}

Illustration 8

Question: **Evaluate the sum : $^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$.**

Solution: Since $^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$ = sum of even terms coefficients in the expansion of $(1 + x)^8$

$= 2^{8-1} = 2^7 = 128.$

- **Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form $k \cdot {}^nC_r$.**

Illustration 9

Question: **Show that $C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \ldots\ldots + n \cdot C_n = n \cdot 2^{n-1}$.**

Solution: The numbers multiplying binomial coefficients are 1, 2, 3, …., n and these are in arithmetic progression.

Let $S = C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \ldots + (n-1)C_{n-1} + n \cdot C_n$

$S = n \cdot C_0 + (n-1)C_1 + (n-2)C_2 + (n-3) \cdot C_3 + \ldots + 1 \cdot C_{n-1}$

(Writing the terms in the reverse order and remembering that $C_r = C_{n-r}$), adding

$$2S = n \cdot C_0 + n \cdot C_1 + n \cdot C_2 + \ldots + n \cdot C_{n-1} + n \cdot C_n$$

$$= n \cdot [C_0 + C_1 + C_2 + \ldots\ldots C_n] = n \cdot 2^n$$

$\therefore \quad S = n \cdot 2^{n-1}$

Illustration 10

Question: **Show that $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2$**

$$= \begin{cases} 0, \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, n!}{\left(\dfrac{n}{2}\right)!\left(\dfrac{n}{2}\right)!}, \text{if } n \text{ is even.} \end{cases}$$

Solution: Consider the product of the expansion of $(1+x)^n$ and $\left(1-\frac{1}{x}\right)^n$ and compare the constant term.

$$C_0^2 - C_1^2 + C_2^2 + ... + (-1)^n C_n^2 = \text{constant term in } (1+x)^n \left(1-\frac{1}{x}\right)^n$$

$$= \text{constant term in } \frac{(1+x)^n\,(x-1)^n}{x^n}$$

$$= \text{constant term in } \frac{(-1)^n (1-x^2)^n}{x^n}$$

$= \text{coefficient of } x^n \text{ in } (-1)^n\,(1-x^2)^n$

= 0, if n is odd since all the terms in $(1-x^2)^n$ contain only even power of x

= coefficient of x^{2m} in $(-1)^{2m}(1-x^2)^{2m}$, if n is even = 2m

$$= (-1)^m\ {}^{2m}C_m = (-1)^m\ {}^{2m}C_m = (-1)^m \frac{(2m)!}{m!\,m!} = \frac{n!\,(-1)^{n/2}}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

Illustration 11

Question: **Show that**

$$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + ... + C_{n-r}\,C_n = \frac{(2n)!}{(n-r)!\,(n+r)!}$$

Solution: Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + C_{r+1}x^{r+1} + ... + C_nx^n$

and $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + ... + \frac{C_n}{x^n}$

In the product of these two expansions, collecting the coefficient of x^r

$$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + ... + C_{n-r}\,C_n = \text{coefficient of } x^r \text{ in } \frac{(1+x)^{2n}}{x^n}$$

$= \text{coefficient of } x^{n+r}\ in\ (1+x)^{2n}$

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n+r)!\,(n-r)!}$$

PRACTICE PROBLEMS

PP1. Using binomial theorem, expand $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$.

PP2. Evaluate : $(3+\sqrt{2})^5-(3-\sqrt{2})^5$.

PP3. Find the 5th term from the end in the expansion of $\left(x+\frac{1}{x}\right)^{12}$.

PP4. Show that the term containing x^3 does not exist in the expansion of $\left(3x-\frac{1}{2x}\right)^8$.

PP5. Show that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ and the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

PP6. Find the coefficient of x^5 in the expansion of $(1+x)^3(1-x)^6$.

PP7. Show that the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

PP8. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}.2^n$.

PP9. Using binomial theorem, prove that (6^n-5n) always leaves the remainder 1 when divided by 25.

PP10. If in the expansion of $(1+x)^n$, the coefficients of pth and qth terms are equal prove that $p+q=n+2$, where $p\neq q$.

PP11. If a, b, c and d in any binomial expansion be the 6th, 7th, 8th and 9th terms respectively, then prove that $\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}$.

PP12. Evaluate $C(8, 0)+C(8, 2)+C(8, 4)+\ldots\ldots+C(8, 8)$.

PP13. Prove that $3C_0-8C_1+13C_2-18C_3+\ldots\ldots\ldots\ldots$ upto $(n+1)$ terms = 0.

PP14. Prove that $C_1+C_2+C_3+\ldots\ldots.+C_n=(1+2+2^2+\ldots..+2^{n-1})$, where C_r denotes the coefficient of x^r in the expansion of $(1+x)^n$.

PP15. Show that $1+(n-1)+\frac{(n-1)(n-2)}{2!}+\frac{(n-1)(n-2)(n-3)}{3!}+\ldots.+1=2^{n-1}$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Show that $C_1 - 2 . C_2 + 3 . C_3 - 4 . C_4 + ...+ (-1)^{n-1}\, n . C_n = 0$.

Solution:

The problem can be done by differentiating the expansion of $(1+x)^n$ and then putting $x = -1$.

Alternative method

L.H.S. $= {}^nC_1 - 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 - .... + (-1)^{n-1} \cdot n \cdot {}^nC_n$

$$= n\left\{1 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 - {}^{n-1}C_3 + ... + (-1)^{n-1}\,{}^{(n-1)}C_{n-1}\right\}$$

$$= n\,(1-1)^{n-1} = n \times 0 = 0$$

***Example* 2:**

Show that $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \dfrac{(2n)!}{n!\,n!}$

Solution:

This example can be solved by considering two binomial expansions $(1+x)^n$ and $\left(1+\dfrac{1}{x}\right)^n$ in which the coefficients of x^n and $\dfrac{1}{x^n}$ are equal and in the product of these expansions the constant term will contain the square of binomial coefficients.

Consider $(1+x)^n = C_0 + C_1\, x + C_2\, x^2 + C_n\, x^n$

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_n}{x^n}$$

Taking the product of these two expansions and collecting the constant term in the product,

Constant term in R.H.S. $= C_0^2 + C_1^2 + C_2^2 + ...\, C_n^2$

= constant term in L.H.S. = constant term in $(1+x)^n \left(1+\dfrac{1}{x}\right)^n$

= constant term in $\dfrac{(1+x)^{2n}}{x^n} =$ coefficient of x^n in $(1+x)^{2n}$

$$= {}^{2n}C_n = \frac{(2n)!}{(n!)\,(n!)}$$

***Example* 3:**

Find the number of terms in the expansion of $(a + b + c)^n$, where n Č N.

Solution:

$(a + (b + c))^n = a^n + {}^nC_1\, a^{n-1} (b + c)^1 + {}^nC_2 a^{n-2}(b + c)^2 + \ldots + {}^nC_n(b + c)^n$

Further expanding each term of R.H.S.,

First term on expansion gives one term

Second term on expansion gives two terms and so on.

$\therefore$ total number of terms = $1 + 2 + 3 + \ldots + (n + 1) = \frac{(n+1)(n+2)}{2}$

***Example* 4:**

Find the number of terms which are free from fractional powers in the expansion of $(a^{1/5} + b^{2/3})^{45}$, a ÷ b.

Solution:

The general term in the expansion of $(a^{1/5} + b^{2/3})^{45}$ is

$T_{r+1} = {}^{45}C_r\, (a^{1/5})^{45-r}\, (b^{2/3})^r = {}^{45}C_r a^{9-(r/5)}\, b^{2r/3}$.

This will be free from fractional powers if both $r/5$ and $2r/3$ are whole numbers i.e. if $r = 0, 15, 30, 45$.

Hence, there are only four terms which are free from fractional powers.

***Example* 5:**

If n is an odd natural number, then find the value of $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$.

Solution:

Now, $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$

$= \sum_{r=0}^{(n+1)/2} \left\{ \frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}} \right\}$ (collecting the terms equidistant from the beginning and end in pairs)

$= \sum_{r=0}^{(n+1)/2} (-1)^r \left\{ \frac{1}{{}^nC_r} + \frac{-1}{{}^nC_r} \right\}$ ($\because (-1)^n = -1$ as n is odd)

$= 0$.

***Example* 6:**

Find the coefficient of x^m in $(1 + x)^r + (1 + x)^{r+1} + (1 + x)^{r+2} + \ldots\ldots + (1 + x)^n$, $r \le m \le n$.

Solution:

Required coefficient = coefficient of x^m in $\frac{(1+x)^r \{(1+x)^{n-r+1} - 1\}}{1+x-1}$

($\because$ given series is a G.P. of $n - (r - 1)$ terms with common ration $1 + x$)

= coefficient of x^{m+1} in $(1 + x)^{n+1} - (1 + x)^r$

$= {}^{n+1}C_{m+1}$ (note than $m + 1 > r$)

***Example* 7:**

If $\frac{1}{\sqrt{2x+1}} \times \left[\left(\frac{1+\sqrt{2x+1}}{2}\right)^n - \left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right]$ is a polynomial of degree 5, then find the value of n.

Solution:

Now $\frac{1}{\sqrt{2x+1}} \times \left\{\left(\frac{1+\sqrt{2x+1}}{2}\right)^n - \left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right\}$

$$= \frac{1}{2^n}\frac{2}{\sqrt{2x+1}} \times \left\{{}^nC_1\sqrt{2x+1} + {}^nC_3(\sqrt{2x+1})^3 + {}^nC_5(\sqrt{2x+1})^5 + \ldots..\right\}$$

$$= \frac{1}{2^{n-1}}\left\{{}^nC_1 + {}^nC_3(2x+1) + {}^nC_5(2x+1)^2 + {}^nC_7(2x+1)^3 + {}^nC_9(2x+1)^4 + {}^nC_{11}(2x+1)^5 + \ldots..\right\}$$

Since, this polynomial is given to be of degree 5, therefore, n can be 11 or 12.

***Example* 8:**

Find the coefficient of (i) x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ (ii) and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$. Find the relation between a and b if these coefficients are equal.

Solution:

The general term in $\left(ax^2 + \frac{1}{bx}\right)^{11} = {}^{11}C_r\ (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r\ \frac{a^{11-r}}{b^r}\ x^{22-3r}$

If in this term, power of x is 7, then 22 – 3r = 7 $\Rightarrow$ r = 5

$\therefore$ coefficient of $x^7 = {}^{11}C_5\ \frac{a^6}{b^5}$ … (i)

The general term in $\left(ax - \frac{1}{bx^2}\right)^{11} = (-1)^r\ {}^{11}C_r\ (ax)^{11-r} \left(\frac{1}{bx^2}\right)^r$

$= (-1)^r\ {}^{11}C_r\ \frac{a^{11-r}}{b^r}\ x^{11-3r}$

If in this term power of x *is* -7, *then* $11 - 3r = -7 \Rightarrow r = 6$

$\therefore$ coefficient of $x^{-7} = (-1)^6\ {}^{11}C_6\ \frac{a^{11-6}}{b^6} = {}^{11}C_5\ \frac{a^5}{b^6}$ … (ii)

If these two coefficients are equal, then $= {}^{11}C_5\ \frac{a^6}{b^5} = {}^{11}C_5\ \frac{a^5}{b^6}$

$\Rightarrow$ $a^6\ b^6 = a^5\ b^5 \Rightarrow a^5\ b^5\ (ab-1) = 0 \Rightarrow ab = 1\,(a \neq 0, b \neq 0)$

***Example* 9:**

If *a*, *b*, *c* , *d* are any four consecutive coefficients of a binomial expansion, prove that

$\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$ **or** $\frac{a+b}{a}, \frac{b+c}{b}, \frac{c+d}{c}$ **are in H.P.**

Solution:

Let a, b, c, d be the coefficient of $(r+1)^{th}, (r+2)^{th}, (r+3)^{th}$ *and* $(r+4)^{th}$ terms of $(1+x)^n$.

$\therefore$ $a = {}^nC_r,\ b = {}^nC_{r+1},\ c = {}^nC_{r+2},\ d = {}^nC_{r+3}$

$$\frac{a}{a+b} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{{}^nC_r}{{}^{(n+1)}C_{r+1}} = \frac{r+1}{n+1}$$

$$\frac{b}{b+c} = \frac{{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{{}^nC_{r+1}}{{}^{(n+1)}C_{r+2}} = \frac{r+2}{n+1}$$

Similarly $\frac{c}{c+d} = \frac{r+3}{n+1}$

Hence $\frac{a}{a+b} + \frac{c}{c+d} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1} = 2 \cdot \frac{b}{b+c}$

$\therefore$ $\frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in A.P.

***Example* 10:**

Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Solution:

$$1 + x + x^2 + x^3 = (1 + x) + x^2\,(1 + x) = (1 + x)\,(1 + x^2)$$

$$\therefore \quad (1 + x + x^2 + x^3)^{11} = (1 + x)^{11}\,(1 + x^2)^{11}$$

$$= \left[\sum_{r=0}^{11} {}^{11}C_r\; x^r\right] \times \left[\sum_{s=0}^{11} {}^{11}C_s\;(x^2)^s\right]$$

The general term in the product of these two series is ${}^{11}C_r \times {}^{11}C_s\; x^{r+2s}$

Now r + 2s must be equal to 4 for values of r, s, $0 \le r, s \le 11$.

The possible values of r and s are r = 0, s = 2; r = 2, s = 1; r = 4, s = 0

$\therefore$ coefficient of $x^4 = {}^{11}C_0 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4 \times {}^{11}C_0 =$ 55 + 605+330 = 990

***Example* 11:**

Find the coefficient of x^{50} in the expansion of

$$(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots\ldots + 1001\,x^{1000}$$

Solution:

Take $(1 + x)^{1000}$ common, and let $\dfrac{x}{1 + x} = r$

$$S = (1 + x)^{1000}\left[1 + 2\,\frac{x}{1 + x} + \frac{3x^2}{(1 + x)^2} + \ldots.. + 1001\left(\frac{x}{1 + x}\right)^{1000}\right]$$

$$= (1 + x)^{1000}\,[1 + 2r + 3r^2 + \ldots. + 1001\,r^{1000}]$$

$$= (1 + x)^{1000}\left\{\left(\frac{1 - r^{1001}}{(1 - r)^2}\right) - \frac{1001\,r^{1000}}{1 - r}\right\}$$, using the formula of A.G.P.

$$= (1 + x)^{1000}\left\{\frac{1 - \left(\dfrac{x}{1 + x}\right)^{1001}}{\left(\dfrac{1}{1 + x}\right)^2} - \frac{1001\left(\dfrac{x}{1 + x}\right)^{1001}}{\dfrac{1}{1 + x}}\right\} = (1 + x)^{1002} - x^{1001}\,(1 + x) - 1001\,x^{1001}$$

$\therefore$ coefficient of x^{50} in S = coefficient of x^{50} in $(1 + x)^{1002}$

$= {}^{1002}C_{50}$

***Example* 12:**

Find the sum of the series

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \text{... upto m terms}\right].$$

Solution:

We have

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \text{... upto m terms}\right]$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r \cdot \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{3}{4}\right)^r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{7}{8}\right)^r + \text{... upto m terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \text{........ upto m terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \text{........ upto m terms}$$

$$= \frac{1}{2^n} + \frac{1}{(2^2)^n} + \frac{1}{(2^3)^n} + \text{........ upto m terms}$$

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right)$$ being the sum of m terms of a G.P. with $r = \frac{1}{2^n}$

$$= \frac{2^{mn}-1}{2^{mn}\,(2^n-1)}$$

***Example* 13:**

Prove that the coefficient of x^r in the expansion

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \dots + (x+2)^{n-1} \text{ is } \left(3^{n-r} - 2^{n-r}\right){}^nC_r.$$

Solution:

The expression $= (x+3)^{n-1}\left\{1 + r + r^2 + \dots + r^{n-1}\right\}$, where $r = \frac{x+2}{x+3}$

$$= (x+3)^{n-1}\left(\frac{1-r^n}{1-r}\right)$$ being the sum of a G.P.

$$= (x+3)^{n-1}\left(\frac{1-\left(\frac{x+2}{x+3}\right)^n}{\left(1-\frac{x+2}{x+3}\right)}\right) = (x+3)^n - (x+2)^n$$

$$= (3+x)^n - (2+x)^n$$

$$\therefore \text{coefficient of } x^r = {}^nC_r\, 3^{n-r} - {}^nC_r\, 2^{n-r} = {}^nC_r\left(3^{n-r} - 2^{n-r}\right)$$

***Example* 14:**

Show that the sum of the product of the C_i's taken two at a time and represented by $\sum\sum_{1 \le i < j \le n} C_i C_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$.

Solution:

We know that $C_0 + C_1 + C_2 + ... + C_n = 2^n$ squaring,

$(C_0 + C_1 + C_2 + ... + C_n)^2 = 2^{2n}$

i.e. $\left(C_0^2 + C_1^2 + ... + C_n^2\right) + 2\sum\sum C_iC_j = 2^{2n}$

$\therefore 2\sum\sum C_iC_j = 2^{2n} - \left\{C_0^2 + C_1^2 + ... + C_n^2\right\}$

$\therefore \sum\sum C_iC_j = 2^{2n-1} - \frac{(2n)!}{2\,(n!)\,(n!)} = 2^{2n-1} - \frac{(2n)!}{2\,(n!)^2}$

***Example* 15:**

Show that $C_1^2 - 2C_2^2 + 3\cdot C_3^2 - ... - (2n)C_{2n}^2 = (-1)^{n-1} n\cdot C_n$ where $C_r = {}^{2n}C_r$

Solution:

Let $S = C_1^2 - 2\cdot C_2^2 + 3\cdot C_3^2 - .... - (2n)\,C_{2n}^2$

$= -(2n)\,C_0^2 + (2n-1)\,C_1^2 - (2n-2)\,C_2^2 - .... + C_1^2,$

since $C_r = {}^{2n}C_r = {}^{2n}C_{2n-r} = C_{2n-r}$

adding

$2S = (-2n)\left\{C_0^2 - C_2^2 - ... + C_{2n}^2\right\}$

$= (-2n)\,(-1)^n \;\; {}^{2n}C_n$

$\therefore S = (-1)^{n+1}\cdot n\cdot {}^{(2n)}C_n = (-1)^{n-1}\cdot nC_n$

EXERCISE – I

1. If the coefficient of x^6 in $\left(x^3+\frac{k}{x}\right)^6$ is 160, find the value of *k*.

2. Find the largest coefficient in the expansion of $(1+x)^{24}$.

3. If $\left(1+x-2x^2\right)^4=1+a_1x+a_2x^2+......+a_8x^8$, then find the value of $a_2+a_4+a_6+a_8$.

4. In the expansion of $(1+x)^{50}$ find the sum of the coefficients of odd powers of *x*.

5. Find the value of sum ${}^{20}C_0+{}^{20}C_1+{}^{20}C_2+....+{}^{20}C_{10}$.

6. Find the greatest integer less than or equal to $\left(\sqrt{2}+1\right)^6$.

7. Find the coefficient of the term independent of *x* in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$.

8. Find the coefficient of x^{10} in the expansion of $\left(1+x^2-x^3\right)^8$.

9. Find the total number of terms in $(x+y)^{200}+(x-y)^{200}$ after simplification.

10. If the binomial coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in A.P., then find the value of *n*.

11. Find the number of terms in the expansions of $\left(1+2x+x^2\right)^{20}$.

12. Find the coefficient of the term independent of *x* in the expansion of $\left(x^{1/6}-x^{-1/3}\right)^9$.

13. The sum of the coefficients in the expansion of $\left(1-2x+5x^2\right)^n$ is '*a*' and the sum of coefficients in the expansion of $(1+x)^{2n}$ is '*b*', then find the relation between '*a*' and '*b*'.

14. Find the sum of the last ten coefficients in the expansion of $(1+x)^{19}$.

15. Find the sum $\frac{1}{2}\,{}^{10}C_0-{}^{10}C_1+2.\,{}^{10}C_2-2^2.\,{}^{10}C_3+......+2^9.\,{}^{10}C_{10}$.

EXERCISE – II

1. Find the coefficient of $a_1^2 a_2 a_3$ in the expansion of $(a_1 + a_2 + a_3)^4$.

2. If the sum of the binomial coefficients in the expansion of $(x + y)^n$ is 1024, then the greatest binomial coefficient occurs in the r^{th} term where r is equal to

3. The value of $1.C_1 + 3.C_3 + 5.C_5 + 7.C_7 + ...$, where $C_0, C_1, C_2, ...C_n$ are binomial coefficients in the expansion of $(1 + x)^n$, is

4. If $a_n = \sum_{r=0}^{n} \frac{1}{^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{^nC_r}$ equals

5. $^nC_n + {}^{(n+1)}C_n + {}^{(n+2)}C_n + ... + {}^{(n+m)}C_n + {}^{(n+m+1)}C_n$ is equal to

6. The expression $\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$ is a polynomial of degree

7. If C_r stands for ^nC_r and p and q are any two numbers, then
$p \cdot C_0 - (p + q)\, C_1 + (p + 2q)\, C_2 - ....\ to\ (n + 1)$ terms is equal to

8. The value of term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ is

9. If $C_0, C_1, ..., C_{2n}$ be the binomial coefficients in the expansion of $(1 + x)^{2n}$, then the value of $C_0 + 2^2 C_2 + 2^4 C_4 + ... + 2^{2n} C_{2n}$ is equal to

10. The value of $^{30}C_0\ {}^{30}C_{10} - {}^{30}C_1\ {}^{30}C_{11} + ... + {}^{30}C_{20}\ {}^{30}C_{30}$ is

11. The coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is

12. Find the sum of coefficients in the expansion of binomial $(5p - 4q)^n$, where n is a positive integer.

13. Show that $a\ {}^nC_0 - (a - 1)\ {}^nC_1 + (a - 2)\ {}^nC_2 - (a - 3)\ {}^nC_3 + \ldots.. (-1)^n (a - n)\ {}^nC_n = 0$, where a and n are positive integers.

14. How many terms are there in the expansion of $((x + y)^4 - (x - y)^4)^6$.

15. Find the term independent of *x* in the expansion of $\left(x^2+\frac{1}{x^2}\right)^{2n}$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $\frac{64x^6}{729}-\frac{32x^4}{27}+\frac{20x^2}{3}-20+\frac{135}{4x^2}-\frac{243}{8x^4}+\frac{729}{64x^6}$

PP2. $1178\sqrt{2}$

PP3. $495x^4$

PP6. –6

PP11. 2^7

EXERCISE – I

1. 2

2. ${}^{24}C_{12}$

3. 7

4. 2^{49}

5. $2^{19}+\frac{1}{2}\frac{(20)!}{(10!)^2}$

6. 198

7. $\frac{5}{4}$

8. 476

9. 101

10. 7

11. 41

12. –84

13. $a = b$

14. 2^{18}

15. $\frac{1}{2}$

EXERCISE – II

1. 12

2. 6

3. $n.2^{n-2}$

4. $\frac{1}{2}na_n$

5. ${}^{(n+m+2)}C_{m+1}$

6. 7

7. 0

8. $\frac{17}{54}$

9. $\frac{3^{2n}+1}{2}$

10. ${}^{30}C_{10}$

11. ${}^{12}C_6+2$

12. 1

14. 7

15. $\frac{1.3.5.....(2n-1)}{n!}2^n$