LESSON 7

PERMUTATIONS & COMBINATIONS

1. INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35. We start this chapter with principle of product or fundamental principle of counting.

2. FUNDAMENTAL PRINCIPLE OF COUNTING

2.1 MULTIPLICATION PRINCIPLE OF COUNTING

If a job can be done in m ways, and when it is done in any one of these ways another job can be done in n, then both the jobs together can be done in mn ways. The rule can be extended to move number of jobs.

Illustration 1

Question: **Find the number of three digit numbers in which all the digits are distinct, odd and number is a multiple of 5.**

Solution: Here it is equivalent to completing three jobs of filling units, tens and hundred place. Now number of ways of filling unit place is only one i.e. 5. Now, four odd digits are left, hence ten's place can be filled in four ways and hundred's place in three ways.

$\therefore$ number of required three digit natural numbers is $1 \times 4 \times 3 = 12$.

Illustration 2

Question: **How many different 7 digit numbers are their sum of whose digits is even?**

Solution: Let us consider 10 successive seven digit numbers

$a_1 a_2 a_3 a_4 a_5 a_6$ 0,

$a_1 a_2 a_3 a_4 a_5 a_6$ 1,

$a_1 a_2 a_3 a_4 a_5 a_6$ 2,

……………………

$a_1 a_2 a_3 a_4 a_5 a_6$ 9,

where a_1 , a_2 , a_3 , a_4 , a_5 , a_6 are some digits. We see that half of these 10 numbers i.e., 5 number have an even sum of digits.

The first digit a_1 can assume 9 different values and each of the digits a_2 , a_3 , a_4 , a_5 , a_6 can assume 10 different values.

The last digit a_7 can assume only 5 different values of which the sum of all digits is even.

$\therefore$ There are $9 \times 10^5 \times 5 = 45 \times 10^5$ seven digit numbers, the sum of whose digits is even.

2.2 ADDITION PRINCIPLE OF COUNTING

If a job can be done in m ways and another job can be done in n ways then either of these jobs can be done in $m + n$ ways. The rule can be extended to more number of jobs.

Illustration 3

Question: **How many three digit numbers *xyz* with $x < y$ and $z < y$ can be formed.**

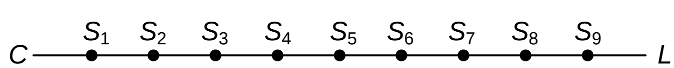
Solution: Obviously $2 \le y \le 9$. If $y = k$, then x can take values from 1 to $k - 1$ and z can take values from 0 to $k - 1$.

Thus required number of numbers $= \sum_{k=2}^{9} (k-1)(k) = 240$.

Illustration 4

Question: **A train is going from Cambridge to London stops at nine intermediate stations. Six persons enter the train during the journey with six different tickets. How many different sets of tickets they have had?**

Solution: For S_1, 9 different tickets available, one for each of the remaining 9 stations; similarly at S_2, 8 different tickets available; and so on.



$\therefore$ Hence, it is clear, that total number of different tickets

$= 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$

Hence, the six different tickets must be any six of these 45 ; and there evidently as many different sets of 6 tickets as there are combinations of 45 things taken 6 at a time tbus we can write if ${}^{45}C_6$.

3. PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a permutation. It is assumed that

- the given things let us say there are n of them are all distinct, that is, no two are alike.
- the arrangement is one of placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement any one thing is used only once. In other words, there is no repetition.

3.1 COUNTING FORMULAE FOR PERMUTATION

To find the value of ${}^nP_r$

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ${}^nP_r$. It must be noted that each space has to be filled up with only one letter.

1	2	3		r

The first space can be filled in n ways. Having filled it, there are n – 1 letters left and therefore the second space can be filled in n – 1 ways. Hence the first two spaces can be filled in $n(n – 1)$ ways. When the first two spaces are filled, there are n – 2 letters left, so that the third space can be filled in n – 2 ways. Therefore the first three spaces can be filled in $n(n – 1)\,(n – 2)$ ways; proceeding like this, the r spaces can be filled in $n(n – 1)\,(n – 2)\,...\,[n – (r – 1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ${}^nP_r$ and its value is equal to

$${}^nP_r = n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$ (using factorial notation $n! = n(n – 1)$ ….. 3.2.1.) where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time = ${}^nP_n = n!$
- ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$
- ${}^nP_r = n\,({}^{n-1}P_{r-1})$ where r = 1, 2, …….. n.

***Illustration* 5**

Question: **Prove from definition that ${}^nP_r = n\,{}^{n-1}P_{r-1}$ and hence deduce the value of ${}^nP_r$.**

1 2 3 r

Solution: Suppose there are n different letters and a row of r blank spaces, each of which has to be filled up with one letter. The number of ways of filling up the blank spaces is the number of ways of arranging n things r at a time, i.e., ${}^nP_r$.

The first space can be filled in n ways. Having filled it, there are $n - 1$ letters left, and $r - 1$ spaces to be filled. By definition, the number of ways of filling up the $r - 1$ spaces with $n - 1$ letters is the number of ways of arranging $n - 1$ things $r - 1$ at a time, i.e. ${}^{n-1}P_{r-1}$.

$\therefore$ the number of ways of filling up the r blank spaces with n different letters is $n \cdot {}^{n-1}P_{r-1}$

i.e., ${}^nP_r = n \cdot {}^{(n-1)}P_{(r-1)}$

$\therefore$ ${}^{n-1}P_{r-1} = (n-1) \cdot {}^{(n-2)}P_{(r-2)}$

$\therefore$ ${}^nP_r = n\,(n-1) \cdot {}^{(n-2)}P_{(r-2)} = n\,(n-1)\,(n-2) \cdot {}^{(n-3)}P_{(r-3)}$

Proceeding like this, we have

$${}^nP_r = n\,(n-1)\,(n-2)\,(n-3) \ldots\ldots (n-r+2) \cdot {}^{(n-r+1)}P_1$$

Note that the last term is formed as ${}^{n-(r-1)}P_{r-(r-1)} = {}^{(n-r+1)}P_1$

Evidently, ${}^{(n-r+1)}P_1 = n - r + 1$

$\therefore$ ${}^nP_r = n\,(n-1)\,(n-2)\,(n-3) \ldots. (n-r+2)\,(n-r+1)$.

In particular,

$${}^nP_0 = 1, {}^nP_1 = n$$

$${}^nP_n = n\,(n-1) \ldots. 1 = n\,!$$

i.e., the number of permutations of n things, taken all at a time is n ! .

Illustration 6

Question: **Show that** ${}^nP_r = {}^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)}$ **.**

Solution: ${}^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)} = \dfrac{(n-1)\,!}{(n-1-r)\,!} + \dfrac{r\,(n-1)\,!}{(n-r)\,!}$

$$= (n-1)\,!\left[\frac{1}{(n-1-r)\,!} + \frac{r}{(n-r)\,!}\right]$$

$= (n-1)!\,\frac{(n-r+r)}{(n-r)!}$ $\quad (\because (n-r)! = (n-r)\,(n-r-1)\,!)$

$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!}$

A common sense interpretation of the identity above is possible. The number of permutations of r things which may be made from n things, $r\,{}^{(n-1)}P_{r-1}$ contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^{n}P_r$.

3.2 IMPORTANT RESULTS

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r\,{}^{n-1}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m\,!)\,(n-m+1)\,!$.
- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!\,(n-m+1)\,!$.

Illustration 7

Question: **Find the number of permutations of *n* different things taken *r* at a time so that two particular things are always included and are together.**

Solution: The two things can be combined as one unit. The remaining $(n-2)$ things may be permuted $(r-2)$ at a time in ${}^{n-2}P_{r-2}$ ways. In each arrangement of these $(r-2)$ things, are created $(r-1)$ spaces in which the unit of two things can be placed. Further the two things in the unit may be interchanged. The number of permutations is

$${}^{n-2}P_{r-2}\cdot(r-1)\cdot 2 = \frac{2(r-1)\,(n-2)\,!}{(n-r)\,!}$$

Illustration 8

Question: **In how many ways can *m* persons entering a theatre, be seated in two rows each containing *n* seats with condition in the first row, no two sit in adjacent seats (2*m* á *n*)?**

Solution: Suppose k persons are seated in the first row where $0 \le k \le m$. In the first row, there are n seats of which $n - k$ are vacant; and the k seats between any two of the vacants have positions in $(n - k + 1)$ spaces.

y The number of seating arrangements for the 1st row = ${}^{(n-k+1)}P_k$

The number of seating arrangements is

$$= \sum_{k=0}^{m} {}^{(n-k+1)}P_k \,.\, {}^{n}P_{m-k}$$

providing for none, one or…. All the m persons being seated in the first row.

3.3 PERMUTATION OF *n* DISTINCT OBJECT WHEN REPETITION IS ALLOWED

- The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

In other words if a job can be completed in n ways and it is to be repeated r number of times then the total number of ways is n^r.

As an example the number of 5 digit numbers, in which digits are not repeated is $9({}^9p_4)$ while when repetition is allowed is 9×10^4.

Also if n distinct things are arrangement at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times \ldots\ldots.. \times n}_{r \text{ times}} = n^r$.

Illustration 9

Question: **There are *m* men and *n* monkeys (*n* > *m*). If a man have any number of monkeys. In how many ways may every monkey have a master?**

Solution: The first monkey can select his master by m ways, and after that the second monkey can select his master again by m ways, so can the third. And so on, hence all monkeys can select master by = $m \times m \times m$ …. up to n times = $(m)^n$ ways.

Illustration 10

Question: **Show that the total number of permutations of *n* different things taken not more than '*r*' at a time, each being allowed to repeat any number of times, is $n\frac{n^r \text{ J } 1}{(n \text{ J } 1)}$.**

Solution: Number of a permutations taken one at a time = n

Number of permutations taken two at a time = n^2

.. ..

Number of permutations taken r at a time = n^r

$\therefore$ Total number of permutations

$$= n + n^2 + n^3 + \ldots\ldots\ldots + n^r$$

$$= \frac{n(n^r - 1)}{(n - 1)}$$

***Illustration* 11**

Question: **In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?**

Solution: The first stall can be occupied by a cow, a horse or a calf, i.e., it can be occupied in 3 ways. The second stall can also be occupied in 3 ways. Therefore the first 2 stalls can be occupied in 3^2 ways. Proceeding like this, it is clear that the 12 stalls can be occupied in 3^{12} ways, i.e., in 531441 ways.

3.4 ARRANGEMENT OF *n* THINGS WHEN ALL ARE NOT DISTINCT

- If given n things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n - (p + q + r)$ are all distinct is given by

$$\frac{n!}{p!\,q!\,r!} \quad (p + q + r \le n)$$

***Illustration* 12**

Question: **Find the number of ways in which we can arrange four letters of the word MATHEMATICS**

Solution: The letters of the word MATHEMATICS are (M, M), (A, A), (T, T), H, E, I, C and S, making eight distinct letters. We can choose four out of them in ${}^8C_4 = 70$ ways, and arrange each of these sets of four in 4! = 24 ways, yielding (70) (24) = 1680 arrangements. Second, we can choose one pair from among the three identical letter pairs, and two distinct letters out of the remaining seven in $({}^3C_1)\,({}^7C_2) = (3)\,[(7 \times 6)/2] = 63$ ways. The letters so obtained can be arranged in 4!/2! = 12 ways, so the number of arrangements in this case is (63) (12) = 756. Finally, we can choose two pairs out of the three identical letter pairs. This can be done in ${}^3C_2 = 3$ ways and the letters obtained can be arranged in 4!/2!2! = 6 ways, so that the number of arrangements in this last case is (3) (6) = 18. Hence the total number of arrangements is 1680 + 756 + 18 = 2454.

4. COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called combinations.

4.1 COUNTING FORMULAE FOR COMBINATIONS

The number of combinations (selections or groupings) that can be formed from *n* different objects taken *r* at a time is denoted by ^nC_r and its value is equal to

$$^nC_r = \frac{n!}{(n-r)!\ r!} \quad (0 \le r \le n)$$

as $$^nC_r = \frac{^nP_r}{r\,!}$$

as in a permutation the arrangement of *r* selected objects out of *n*, is done in *r*! ways and in combination arrangement in a group is not considered.

In particular

- $^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or to select all objects out of *n* distinct objects.
- $^nC_1 = n$ There are *n* ways to select one thing out of *n* distinct things.
- $^nC_r = {}^nC_{n-r}$

  Therefore $^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If *n* is odd then the greatest value of ^nC_r is $^nC_{\frac{n+1}{2}}$ or $^nC_{\frac{n-1}{2}}$.
- If *n* is even then the greatest value of ^nC_r is $^nC_{n/2}$.

Illustration 13

Question: **Prove that product of *r* consecutive positive integer is divisible by *r* ! .**

Solution: Let *r* consecutive positive integers be $(m), (m + 1), (m + 2), \ldots, (m + r - 1)$, where $m \in \mathbf{N}$.

$\therefore$ Product $= m(m + 1)(m + 2) \ldots (m + r - 1)$

$$= \frac{(m-1)!\ m(m+1)\ (m+2) \ldots (m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = r\,!\,.\,\frac{(m+r-1)!}{r\,!\,(m-1)!}$$

which is divisible by *r* ! $\quad (\because {}^{m+r-1}C_r$ is natural number)

Illustration 14

Question: **Show that the number of triangles whose angular points are at the vertices of a given polygon of *n* sides but none of whose sides are the sides of the polygon is $\frac{n\,(n-4)\,(n-5)}{6}$.**

Solution: For any triangle to be possible, 3 of the *n* vertices are to be chosen. This can be done in ${}^{n}C_3$ ways. Of these there are *n* triangles with two sides as adjacent sides of the polygon – like side 1 and side 2; side 2 and side 3 etc, the third side of the triangle being the corresponding diagonal; and there are, with one side of the polygon as a side of the triangle, $(n-4)$ triangles.

∴ Required number of triangles = ${}^{n}C_3 - n - n\,(n-4)$

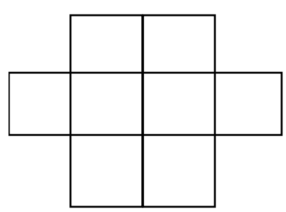
$$= \frac{n(n-1)\,(n-2)}{6} - n - n(n-4)$$

$$= \frac{n}{6}\,\{n^2 - 3n + 2 - 6 - 6n + 24\}$$

$$= \frac{n}{6}\,(n^2 - 9n + 20) = \frac{n(n-4)\,(n-5)}{6}$$

Illustration 15

Question: **Six *X*'s have to be placed in spaces on the adjoining Figure so that each row contains at least one *X*. In how many different ways this can be done?**



Solution: If there be no restriction on the placing of the *X*'s, the number of ways is ${}^{8}C_6 = {}^{8}C_2 = 28$. Of these there are two ways in which the *X* ;s can be placed; one, with the first row empty and the other with the third row empty. These two cases only do not satisfy the condition.

∴ the number of ways = 28 – 2 = **26**.

4.2 IMPORTANT RESULTS OF COMBINATIONS (SELECTIONS)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.

Illustration 16

Question: **A lady desires to give a dinner party for 8 guests. In how many ways can the lady select guests for the dinner from her 12 friends, if two of the guests will not attend the party together?**

Solution: The following three methods of approach are indicated.

(i) Number of ways of forming the party

$= {}^{12}C_8 - {}^{10}C_6$ since ${}^{10}C_6$ is the number of ways of making up the party with both the specified guests included.

$= 495 - 210 = 285$

(OR)

(ii) Number of ways of forming the party

= Number of ways of forming without both of them

+ Number of ways of forming with one of them and without the other

$= {}^{10}C_8 + 2.{}^{10}C_7 = 45 + 240 = 285$

(OR)

(iii) Split the number of ways of forming the party

= those with one of the two (say A) + those without A

$= {}^{10}C_7 + {}^{11}C_8 = 120 + 165 = 285$

- The number of ways in which r objects can be selected from n objects if m particular objects are identical is $\sum_{r=0}^{r} {}^{n-m}C_r$ or $\sum_{r=r-m}^{r} {}^{n-m}C_r$ according as $r \le m$ or $r > m$.

Illustration 17

Question: **A bag contains 23 balls in which 7 are identical. Then find the number of ways of selecting 12 balls from bag.**

Solution: Here $n = 23$, $p = 7$, $r = 12$ $(r > p)$

Hence, required number of selections = $\sum_{r=5}^{12} {}^{16}C_r$

$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$

$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$

$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad (\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad (\because {}^nC_r = {}^nC_{n-r})$

$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$

$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$

5. SELECTION FROM DISTINCT/IDENTICAL OBJECTS

5.1 SELECTION FROM DISTINCT OBJECTS

- The number of ways (or combinations) of selection from n distinct objects, taken at least one of them is

  $${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$$

  Logically it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on ……. and by addition principle of counting the total number of ways of doing either of the job is ${}^nC_1 + {}^nC_2 + ….. + {}^nC_n$

  Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected = $2^n - 1$.

Illustration 18

Question: **Find the number of ways in which we can put *n* distinct objects into two identical boxes so that no box remains empty.**

Solution: Let us first label the boxes 1 and 2. We can select at least one or at most $(n–1)$ balls for box 1 in

${}^nC_1 + {}^nC_2 + ……… {}^nC_{n-1}$ ways

$= {}^nC_0 + {}^nC_1 + ……… {}^nC_n - {}^nC_0 - {}^nC_n$

$= 2^n - 2$

$= 2(2^{n-1} - 1)$ ways.

In this way box 2 is not empty. But since the boxes are identical the number of ways that no box remains empty is $\frac{1}{2} \times 2(2^n - 1) = 2^{n-1} - 1$.

Alternative solutions:

Let us first label the boxes 1 and 2. There are then to choices for each of the *n* objects; we can put it in the first box or in the second box. Therefore the number of choices for *n* distinct objects is $\underbrace{2 \times 2 \times .... \times 2}_{n \text{ times}} = 2^n$.

Two of these choices correspond to either the first or the second box being empty. Thus there are $2^n - 2$ ways in which neither box is empty. If we now remove the labels from the boxes so that they become identical, this number must be divided by 2, yielding the answer $1/2\ (2^n - 2) = 2^{n-1} - 1$.

***Illustration* 19**

Question: **Given five different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green, one blue dye?**

Solution: Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included = $2^5 - 1 = 31$

(1 is subtracted to correspond to the case when none of the green dyes is chosen.)

A similar argument may be advanced in respect of other two colours also.

Number of combinations = $(2^5 - 1)\ (2^4 - 1)\ (2^3)$.

$= 31 \times 15 \times 8 = 3720$

PRACTICE PROBLEMS

PP1. A conference hall has 10 gates. In how many ways can a person enter the hall through one gate and exit through different gate?

PP2. How many different signals can be made by 5 flags from 8 flags of different colors.

PP3. How many words can be formed using letters of word "BRILLIANT"?

PP4. A child has 5 pockets and 4 marbles. In how many ways can the child put the marbles in his pockets?

PP5. Find the domain of definition of the function $f(x) = {}^{7-x}P_{x-3}$ and the set of its values.

PP6. How many words can be formed out of the letters of the word COURAGE so that the vowels are in the odd places?

PP7. How many different six digit numbers are there the sum of whose digits is odd?

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Then find the number of words which have at least one letter repeated

***Solution*:**

Suppose we have 5 places, each of which is to be filled by one letter from the 10 letters. The first place may be filled in 10 ways. When repetitions of the letters are allowed, the second place may also be filled in 10 ways.

Proceeding in this way, it is clear that words with five letters are formed in 10^5 ways. These 10^5 ways also include the number of ways of forming words with all different letters without repetition. These are ${}^{10}P_5$ in number.

$\therefore$ the number of words which have at least one letter repeated is

$$10^5 - {}^{10}P_5 = 100000 - 30240 = 69760$$

***Example* 2:**

Find the value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{(52-j)}C_3$.

***Solution*:**

The given expression = $({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 = ({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= ({}^{49}C_4 + {}^{49}C_3) + {}^{50}C_3 + {}^{51}C_3 = ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3$

$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$, by using the formula ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ repeatedly.

***Example* 3:**

In a plane there are 37 straight lines, of which 13 pass through the point *A* and 11 pass through the point *B*. Besides, no three lines pass through one point, no line passes through both points *A* and *B*, and no two are parallel. Then find the number of intersection points the lines.

***Solution*:**

In the general position, 37 straight lines have ${}^{37}C_2$ points of intersection. But 13 straight lines passing through the point *A* yield one intersection point instead of ${}^{13}C_2$ and 11 straight lines passing through the point *B* yield one intersection point instead of ${}^{11}C_2$.

$\therefore$ the lines have ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ points of intersection.

i.e., 666 – 78 – 55 + 2, i.e., 535.

***Example* 4:**

A set contains (2*n* + 1) elements. The find the number of subsets of the set which contain at most *n* elements.

***Solution*:**

The number of subsets of the set which contain at most *n* elements is

$^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots.. + {}^{2n+1}C_n = N$ (say)

We have $2N = 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots.. + {}^{2n+1}C_n)$

$= ({}^{2n+1}C_0 + {}^{2n+1}C_{2n+1}) + ({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + \ldots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})$ $(\because {}^nC_r = {}^nC_{n-r})$

$= {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots.. + {}^{2n+1}C_{2n+1} = 2^{2n+1} \Rightarrow N = 2^{2n}$

***Example* 6:**

Ten persons are arranged in a row. Find the number of ways of selecting four persons so that no two persons sitting next to each other.

***Solution*:**

To each selection of 4 persons we associated binary sequence of the form 1001001010 where 1(0) at i^{th} place means the i^{th} person is selected (not selected).

There exists one-to-one correspondence between the set of selections of 4 persons and set of binary sequence containing 6 zeros and 4 ones.

We are interested in the binary sequences in which no 2 ones are consecutive. We first arrange 6 zeros.

0 0 0 0 0 0

This can be done in just one way. Now, 4 ones can be arranged at any of the 4 places marked with a cross in the following arrangement.

× 0 × 0 × 0 × 0 × 0 × 0.

We can arrange 4 1's at 7 places in $^7C_4 = 35$ ways.

***Example* 8:**

Find the number of times of the digits 3 will be written when listing the integers from 1 to 1000.

***Solution*:**

Since 3 does not occur in 1000, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form *xyz* where $0 \le x, y, z \le 9$. Let us first count the numbers in which 3 occurs exactly once. Since 3 can occur at one place in $^3C_1$ ways, there are $^3C_1 (9 \times 9) = 3 \times 9^2$ such numbers.

Next, 3 can occur exactly at two places in $({}^3C_2)(9) = 3 \times 9$ such numbers.

Lastly, 3 can occur in all three digits in one number only.

Hence the number of times 3 occurs is

$$1 \times (3 \times 9^2) + 2 \times (3 \times 9) + 3 \times 1 = 300$$

***Example* 9:**

Find the number of ways of arranging six persons (having *A*, *B, C* and *D* among them) in a row so that *A*, *B*, *C* and *D* are always in order *ABCD* (not necessarily together).

***Solution*:**

The number of ways of arranging *ABCD* is 4!. For each arrangement of *ABCD*, the number of ways of arranging six persons is same.

Hence required number is $\frac{6!}{4!} = 30$

***Example* 10:**

Let *S* be the set of all functions from the set *A* to the set *A*. If $n(A) = k$, then find $n(S)$.

***Solution*:**

Each element of the set *A* can be given the image in the set *A* in *k* ways. So, the required number of functions,

i.e., $n(S) = k \times k \times \ldots$ (k times) $= k^k$.

***Example* 11:**

Let *A* be the set of 4-digit numbers $a_1a_2\,a_3a_4$ where $a_1 > a_2 > a_3 > a_4$, then find the value of $n(A)$.

***Solution*:**

Any selection of four digits from the ten digits 0, 1, 2, 3, … 9 gives one such number. So, the required number of numbers = ${}^{10}C_4 = 210$.

***Example* 12:**

15 persons, amongst whom are *A*, *B* and *C* are to speak at a function. Find in how many ways can the speech be done if *A* wants to speak before B and B is to speak before *C* ?

Solution:

We can select three position out of 15 position by ${}^{15}C_3$ ways.

We can provide these position to *A, B, C* by only one ways.

Other 12 persons can speak in 12 ! ways.

Hence total number of ways will be ${}^{15}C_3 \times 1 \times 12\,!$

***Example* 13:**

A committee of twelve is to be formed from 9 women and 8 men. In how many ways can this be done if at least five women have to be included in the committee? In how many of these committees (i) the women are in a majority? (ii) the men are in a majority?

Solution:

The possible ways of formation of the committee are listed:

Constitution of the committee			**Number of ways of formation**
(9) Women	(8) Men		
5	7	$\longrightarrow$	${}^9C_5 \cdot {}^8C_7 = 1008$
6	6	$\longrightarrow$	${}^9C_6 \cdot {}^8C_6 = 2352$
7	5	$\longrightarrow$	${}^9C_7 \cdot {}^8C_5 = 2016$
8	4	$\longrightarrow$	${}^9C_8 \cdot {}^8C_4 = 630$
9	3	$\longrightarrow$	${}^9C_9 \cdot {}^8C_3 = 56$
		Total number of ways	6062

(i) Number of committees with women majority

= 2016 + 630 + 56 = 2702

(ii) Number of committees with men majority = 1008

***Example* 14:**

How many seven digit numbers can be formed using only the three digits 1, 2 and 3; the digit 2 occurring only twice in each number.

Solution:

Any two of the seven digits can be chosen, and in these places, 2 is filled and rest five are filled with 1 or 3.

y The required number is ${}^7C_2 \cdot 2^5 = 672$

***Example* 15:**

There are n points in a plane, no three of which are collinear except '*p*' points all of which are on a line How many (i) straight lines can be formed (ii) triangles can be formed out of these *n* points?

Solution:

(i) To form a line we need two points; and these two points may be chosen in ${}^nC_2$ ways; but, it happens that '*p*' of the '*n*' points are on a line; consequently these points would form only one line instead of ${}^pC_2$.

y Number of lines = ${}^nC_2 - {}^pC_2 + 1$

(ii) Number of triangles = ${}^nC_3 - {}^pC_3$

EXERCISE – I

1. Find *n,* if $(n + 2)! = 2550 \times n!$.

2. How many three letter words can be formed using *a, b, c, d*, *e* if:

(i) repetition is not allowed (ii) repetition is allowed.

3. In how many ways can four jobs I, II, III and IV be assigned to four persons *A, B*, *C* and D if one person is assigned only one job and all are capable of doing each job.

4. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed ?

5. In how many ways 8 beads can be arranged to form a necklace ?

6. How many words can be formed from the letters of the word 'SERIES' which start with *S* and end with *S ?*

7. Among 36 teachers in a college, one principal, one vice-principal and three teacher-incharge are to be appointed. In how many ways can this be done ?

8. The English alphabet has 5 vowels and 21 consonant. How many words with two different vowels and two different consonants can be formed from the alphabets?

9. In how many ways can 6 Hindus and 6 Muslims sit around a round table so that two Muslims may never sit together ?

10. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition) ?

11. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times ?

12. In how many ways can the letters of the word 'STRANGE' be arranged so that

(i) the vowels come together ?

(ii) the vowels never come together ?

(iii) the vowels occupy only the odd places ?

13. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees,

(i) a particular professor is included. (ii) a particular student is included.

(iii) a particular student is excluded.

14. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.

15. Three distinct dice are rolled. Find the number of possible outcomes in which at least one die shows 5.

16. In a group there are 4 girls and 6 boys. If all the 4 girls sit together, find number of ways in which they can be arranged around a table.

17. How many numbers can be formed by using the digits 2, 2, 3, 3, 3, 4, 4 all at a time?

18. There are 15 points in a plane, no three of which are collinear. Find the number of triangles formed by joining them.

19. In how many ways can we select a cricket eleven from 17 players, in which only 5 players can bowl and if each cricket eleven must include exactly 4 bowlers ?

20. In how many ways can 5 red and 5 white balls be drawn from an urn containing 8 red and 6 white balls?

EXERCISE – II

1. If repetition of digits is not allowed then find out how many numbers of four digits divisible by 5 can be formed with the digits 0, 4, 5, 6, 7.

2. A servant has to post 5 letters and there are 4 letter boxes. Find out in how many ways can he post the letters.

3. In how many ways 5 delegates can be put in 6 hotels of a city if there is no restriction.

4. In a dinner party there are 10 Indians, 5 Americans and 5 Englishmen. Find out in how many ways can they be arranged in a row so that all persons of the same nationality sit together.

5. In a class of students there are 4 girls and 6 boys. Find out in how many ways can they be seated in a row so that all the four girls are not together.

6. Find out the number of different arrangements (permutations) of the letters of the word ‘Banana’.

7. How many different words can be formed with the letters of the word “MATHEMATICS”.

8. How many words can be formed out of the letters of the word ‘Article’ so that the vowels occupy the even places.

9. Find out the number of arrangements of the letters of the word ‘Delhi’ if *e* always comes before *i* .

10. If ${}^{n}C_6 : {}^{n-3}C_3 = 33 : 4$, then find out value of *n*.

11. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$, then find out minimum value of *n.*

12. There are *n* stations on a railway line. The number of kinds of tickets printed (no return tickets) is 105, then find out the number of stations.

13. A polygon has 44 diagonals. Find out the number of sides.

14. Find out the sum of all the 4 digits numbers formed with the digits 1, 3, 3, and 0.

15. Find out the number of ways in which six ‘+’ and four ‘–‘ signs can be arranged in a line such that no two ‘–‘ signs occur together.

ANSWERS

EXERCISE – I

CBSE PROBLEMS

1. 49

2. (i) 60 (ii) 125

3. 24

4. 720

5. $\frac{1}{2}(7!)$

6. 12

7. 7539840

8. 50400

9. 86400

10. 11

11. 215

12. (i) 1440 (ii) 3600 (iii) 1440

13. 51300 (i) 10260 (ii) 7695 (iii) 43605

14. (i) 9 (ii) 104

15. 91

16. 6! 4!

17. $\frac{7!}{2!\,3!\,2!}$

18. 455

19. 3960

20. $^{8}C_{5} \cdot {}^{6}C_{5}$

EXERCISE – II

1. 42

2. 4^5

3. 6^5

4. 3 ! 10 ! 5 ! 5 !

5. 10 ! –7 ! 4 !

6. 60

7. $\frac{11!}{2!\,2!\,2!}$

8. 144

9. 60

10. 11

11. 7

12. 15

13. 11

14. 22554

15. 35

ANSWERS TO PRACTICE PROBLEMS

PP1. 90 ways

PP2. $^{8}P_{5}$

PP3. $\frac{9!}{2!\ 2!}$

PP4. 5^4

PP5. Domain {3, 4, 5}

Range {1, 3, 2}

PP6. 144

PP7. 450000