LESSON 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS, INEQUATIONS & EXPRESSIONS

1. INTRODUCTION

Whenever $\sqrt{x}$ is thought of to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide for the existence of a value for $\sqrt{x}$ when $x < 0$. To make this possible we extend the number system so as to include and cover yet another class of numbers, called imaginary numbers.

Let us take the quadratic equation $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$ i.e., $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers.

It is therefore, the symbol i, is thought of to possess the following properties:

(i) It combines with itself and with real numbers satisfying the laws of algebra.

(ii) Whenever we come across –1 we may substitute i^2.

In the light of the foregoing the roots of the equation discussed earlier may be taken as $1 + 3i$, $1 - 3i$.

It is taken that 1 is real part and 3(or – 3) is the imaginary part of this complex number $1 + 3i$ or $1 - 3i$ respectively.

***Illustration* 1**

Question: **If $x = -5 + 2\sqrt{J\ 4}$ find the value of x^4 H $9x^3$ H $35x^2$ J x H 4.**

***Solution*:** $x = -5 + 4i \ (i = \sqrt{-1})$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 + 9x^3 + 35x^2 - x + 4 = (x^2 + 10x + 41)\,(x^2 - x + 4) - 160$ and $x^2 + 10x + 41 = 0$

Hence given expression $= 0 - 160 = -160$

2. COMPLEX NUMBERS

A complex number, represented by an expression of the form $x + iy$ (x, y are real), is taken to be an ordered pair (x, y) of two real numbers, combined to form a complex number and an algebra is defined on the set of such numbers, represented by an ordered pair (x, y) to satisfy the following:

(addition) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(subtraction) $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$

(multiplication) $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

(division) $(x_1, y_1) \div (x_2, y_2) = \left(\dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \dfrac{x_2y_1 + x_1y_2}{x_2^2 + y_2^2}\right)$

For any real number $\alpha, \alpha(x, y) = (\alpha x, \alpha y)$ and if $(x, y) = (x', y')$ then it must be $x' = x$; $y' = y$. In other words, the representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number it is not possible to have two different ordered pairs form of representation. In the light of the foregoing it may be said that the two representation (x, y) – in the ordered pair form and $x + iy$ are indistinguishable.

***Illustration* 2**

Question: **Find the sum and product of the two complex numbers Z_1 Z 2 H 3*i* and Z_2 Z J 1 H 5*i***

Solution: $Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8i$

$Z_1Z_2 = (2 + 3i)\,(-1 + 5i) = -2 + 15i^2 - 3i + 10i = -17 + 7i \quad (i^2 = -1)$

Based on the above discussion we are listing a few points:

1. If $z = a + ib$, then real part of z = Re (z) = a and Imaginary part of z = *Im* (z) = b.
2. If Re (z) = 0, the complex number is purely imaginary.

3. If Im (z) = 0, the complex number is real.
4. The complex number 0 = 0 + 0i is both purely imaginary and real.
5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
6. There is no order relation between complex numbers i.e. $(a + ib) >$ or $< (c + id)$ is a meaningless expression.

Illustration 3

Question: **Express $\frac{1}{(1 - \cos\theta + i\sin\theta)}$ in the form $a + ib$.**

Solution:

$$\frac{1}{(1-\cos\theta + i\sin\theta)} = \frac{(1-\cos\theta) - i\sin\theta}{(1-\cos\theta+i\sin\theta)(1-\cos\theta - i\sin\theta)}$$

$$= \{(1-\cos\theta) - i\sin\theta\}/\{(1-\cos\theta)^2 + \sin^2\theta\} = \frac{(1-\cos\theta) - i\sin\theta}{2 - 2\cos\theta}$$

$$= \frac{1-\cos\theta}{2(1-\cos\theta)} - \frac{i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2} - i\cdot\cot\frac{\theta}{2}$$

3. REPRESENTATION OF A COMPLEX NUMBER

3.1 GEOMETRICAL REPRESENTATION

It is known, from coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.

It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

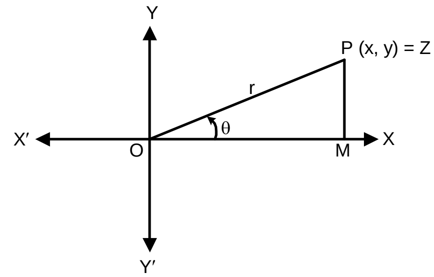
It is therefore, that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be said to be geometrical representation of Z. This association is a bijection – in the mapping language – whereby, this correspondence between Z and P is ONE-ONE and ONTO. It is therefore possible to go over to a point from Z, or reversing the roles, come back to Z from the point.

3.2 ARGAND DIAGRAM

The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane. In this representation, all complex numbers like (2, 0), (3, 0), (–1, 0), (α, 0) with imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number (α, 0), all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly all purely imaginary numbers (with real part 0) like (0, 1), (0, 2), (0, – 3), (0,β) will be marked on the y-axis. Hence the y-axis is also called the imaginary axis in this context. The Cartesian plane (two dimensional plane) is also called the complex plane.

3.3 POLAR REPRESENTATION

Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.



Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$

Thus $\quad z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

where $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ by eulers formula

Thus $\quad z = r(\cos\theta + i\sin\theta)$ can be written as

$\boldsymbol{z = re^{i-}}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2+y^2}$, a non-negative number. $|z| = 0$ for the only number (0, 0).

Illustration 4

Question: **Represent the given complex numbers in polar form:**

(i) (1 H $i\sqrt{3}$)^2 / 4i(1 J $i\sqrt{3}$) (ii) sin ~ – i cos ~ (~ acute) (iii) 1 H cos $\frac{®}{3}$ H i sin $\frac{®}{3}$

Solution: (i) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \frac{(1+i\sqrt{3})^2}{4(\sqrt{3}+i)} = \frac{-2+2i\sqrt{3}}{4(\sqrt{3}+i)} = \frac{(-1+i\sqrt{3})\,(\sqrt{3}-i)}{2(\sqrt{3}+i)\,(\sqrt{3}-i)} = \frac{-\sqrt{3}+\sqrt{3}+4i}{2(3+1)} = \frac{i}{2}$$

and $\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Hence $\frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}\,e^{i\pi/2}$

(ii) Real part > 0; Imaginary part < 0

argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle.

$$\therefore \sin\alpha - i\cos\alpha = \cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right) = e^{i\left(\alpha - \frac{\pi}{2}\right)}$$

(iii) $1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 2\cos^2\frac{\pi}{6} + i\cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6}e^{i\pi/6}$$

4. CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

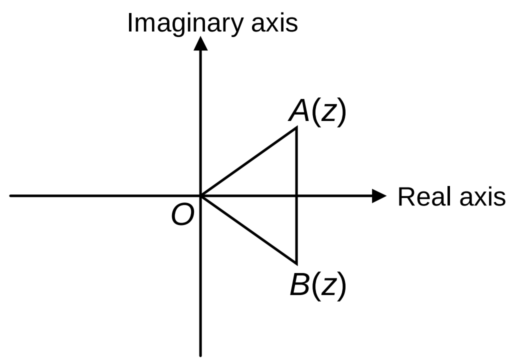
Note that, sum = $(a + ib) + (a - ib) = 2a$ which is real

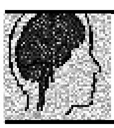
and product $= (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$

$= a^2 - (-1)\, b^2 = a^2 + b^2$ which is real.

4.1 PROPERTIES OF CONJUGATE

- $(\bar{z}) = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ $(z_2 \neq 0)$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(\overline{z_1}z_2) = 2\text{Re}(z_1\overline{z_2})$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\overline{z_1})$





If $|z| = 1$ can we say $\bar{z} = \dfrac{1}{z}$?

5. MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

5.1 PROPERTIES OF MODULUS

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1z_2| = |z_1||z_2|$
  In general $|z_1z_2z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots\ldots|z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$
  In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z_1 \pm z_2| \geq ||z_1| - |z_2||$
  In particular, if $|z_1 - z_2| = ||z_1| - |z_2||$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z^n| = |z|^n$
- $||z_1| - |z_2|| \leq |z_1| + |z_2|$
  Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$
- $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$ or
  $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$
- $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in \mathbf{R}$.
- Unimodular : i.e., unit modulus
  If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in \mathbf{R}$.

  Note: $\dfrac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

Some of the proofs are given as:

õ $|Z_1 Z_2| = |Z_1| \times |Z_2|$

Proof:

Let $Z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $Z_1 Z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} = r(\cos\theta + i\sin\theta)$,

where $r = r_1 r_2$ and $\theta = \theta_1 + \theta_2$.

$\therefore |Z_1 Z_2| = r = r_1 r_2 = |Z_1| \times |Z_2|$

õ $|Z_1 Z_2 \ldots Z_n| = |Z_1| \times |Z_2| \times |Z_3| \times \ldots \times |Z_n|$

Proof follows by writing $Z_1 Z_2 \ldots Z_n$ as the product of $Z_1 Z_2 \ldots Z_{n-1}$ and Z_n and applying property (1) repeatedly.

õ $|Z^n| = |Z|^n$

Proof follows if we take $Z_1 = Z_2 = Z_3 = \ldots = Z_n$

õ $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$

$$\left(\text{since } \frac{1}{\cos\theta_2 + i\sin\theta_2} = \cos\theta_2 - i\sin\theta_2\right)$$

$$= \left(\frac{r_1}{r_2}\right)\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

Hence $\left|\frac{Z_1}{Z_2}\right| = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$

õ First triangle inequality $|Z_1| + |Z_2| \geq |Z_1 + Z_2|$

Proof:

$|Z_1 + Z_2| = |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)|$

$$= |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$$

$$= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2}, \text{ since } \cos(\theta_1 - \theta_2) \leq 1$$

$\therefore |Z_1 + Z_2| \leq \sqrt{(r_1 + r_2)^2}$

or $|Z_1 + Z_2| \leq r_1 + r_2$. Thus $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$.

Note: Equality occurs only when $\theta_1 = \theta_2$ i.e. when Z_1 and Z_2 have the same amplitude.

õ Second triangle inequality

$|Z_1 - Z_2| \geq |Z_1| \sim |Z_2|$

Proof

$$Z_1 - Z_2 = r_1\cos\theta_1 - r_2\cos\theta_2 + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$

$$\therefore |Z_1 - Z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1r_2}\text{, since } \cos(\theta_1 - \theta_2) \leq 1$$

$$\therefore |Z_1 - Z_2| \geq \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|$$

$$|Z_1 - Z_2| \geq r_1 \sim r_2 = |Z_1| \sim |Z_2|.$$

õ $|\overline{Z}| = |Z|$

Proof:

$| Z | = \sqrt{x^2 + y^2}$ if $Z = x + iy$

Then $\overline{Z} = x - iy$

$$\therefore |\overline{Z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$$

$$\therefore |\overline{Z}| = |Z|$$

***Illustration* 5**

Question: **If $|z$ J $2 + i|$ á 2 then find the greatest and least value of $|z|$.**

Solution: Given that

$|z - 2 + i| \leq 2$ …(i)

$\because$ $|z - 2 + i| \geq ||z| - |2 - i||$

$\therefore$ $|z - 2 + i| \geq ||z| - \sqrt{5}|$ …(ii)

From (i) and (ii)

$||z| - \sqrt{5}| \leq |z - 2 + i| \leq 2$

$\therefore$ $||z| - \sqrt{5}| \leq 2$

$\Rightarrow \quad -2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow \quad \sqrt{5} - 2 \le |z| \le \sqrt{5} + 2$

Hence greatest value of |z| is $\sqrt{5}+2$ and least value of |z| is $\sqrt{5}-2$.

***Illustration* 6**

Question: **If Z_1 and Z_2 be two complex numbers such that $\left|\dfrac{Z_1 - 2Z_2}{2 - Z_1\bar{Z}_2}\right| = 1$ and $|Z_2| \div 1$. What is the value of $|Z_1|$?**

Solution: $|Z_1 - 2Z_2| = |2 - Z_1\bar{Z}_2|$

$\therefore |Z_1 - 2Z_2|^2 = |2 - Z_1\bar{Z}_2|^2$

$\therefore (Z_1 - 2Z_2)(\bar{Z}_1 - 2\bar{Z}_2) = (2 - Z_1\bar{Z}_2)(2 - \bar{Z}_1 Z_2)$

$\therefore Z_1\bar{Z}_1 - 2\bar{Z}_1 Z_2 - 2Z_1\bar{Z}_2 + 4Z_2\bar{Z}_2 = 4 - 2Z_1\bar{Z}_2 - 2\bar{Z}_1 Z_2 + Z_1\bar{Z}_1 Z_2\bar{Z}_2$

$\therefore Z_1\bar{Z}_1 + 4Z_2\bar{Z}_2 - 4 - Z_1\bar{Z}_1 Z_2\bar{Z}_2 = 0$

$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2|Z_2|^2 - 4 = 0$ i.e. $(|Z_1|^2 - 4)(|Z_2|^2 - 1) = 0$

Since $|Z_2| \ne 1$ it is that $|Z_1|^2 = 4$ i.e. $|Z_1| = 2$

6. ARGUMENT OF A COMPLEX NUMBERS

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$.

θ is called the argument of Z or amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \le \pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a + ib$ is given by α, $\pi-\alpha$, $-\pi+\alpha$, or $-\alpha$ if $a + ib$ is in first, second, third or fourth quadrant respectively, where $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right|$.

For example

- Z = 1 + i = (1, 1) and is marked by point P(1, 1) lies in first quadrant.
  $\therefore$ |Z| = $\sqrt{2}$ and arg $Z = \pi/4$.

- If $Z = 1 - i = (1, -1)$, then P lies in the fourth quadrant and $|Z| = \sqrt{2}$ and arg $Z = -\pi/4$.
- If $Z = -1 + i = (-1, 1)$, then P lies in the second quadrant and arg $Z = \frac{3\pi}{4}$.
- If $Z = -1 - i$, arg $Z = -\frac{3\pi}{4}$.
- Argument of all positive real numbers like 1, 2, 3, $\frac{1}{2}$, … is 0 since they are marked on the positive x-axis. Argument of all negative real numbers like – 1, – 2, – 3, … is π since they are marked on OX′. Argument of purely imaginary numbers like i, $2i$, $3i$, … is $\frac{\pi}{2}$ since these are marked on the positive y-axis. Argument of purely imaginary numbers like $-i$, $-2i$, $-3i$, … is $-\frac{\pi}{2}$.

***Illustration* 7**

Question: **Among the complex numbers *z* which satisfies |*z* J 25*i* | á 15, find the complex numbers *z* having**

(i) least positive argument **(ii) maximum positive argument**

(iii) least modulus **(iv) maximum modulus**

Solution: The complex numbers *z* satisfying the condition |*z* J 25*i*| ≤ 15 are represented by the points inside and on the circle of radius 15 and centre at the point *C*(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.

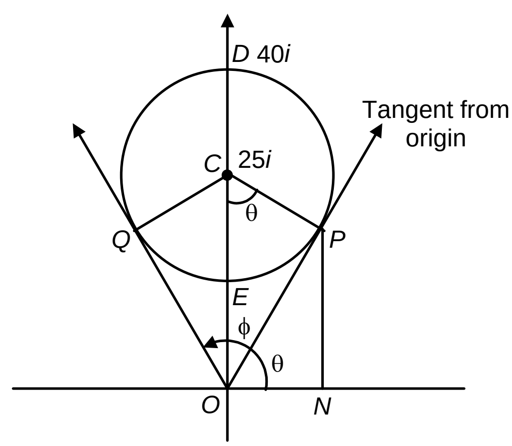
Here θ = least positive argument and ϕ = maximum positive argument

$\therefore$ In ΔOCP, $OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{4}{3}$

$\Rightarrow$ $\theta = \tan^{-1}\left(\frac{4}{3}\right)$



Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore Z_P = 20(\cos\theta + i\sin\theta) = 20\left(\frac{3}{5} + i\,\frac{4}{5}\right)$

$\therefore$ $Z_P = 12 + 16i$

Similarly $Z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $Z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $Z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

6.1 PROPERTIES OF ARGUMENTS

- Arg (z_1z_2) = Arg (z_1) + Arg (z_2) + $2k\pi$ ($k = 0$ or 1 or -1)
  In general Arg $(z_1z_2z_3 \ldots z_n)$ = Arg (z_1) + Arg (z_2) + Arg (z_3) + …… + Arg (z_n) + $2k\pi$ (where $k \in I$)
- Arg $\left(\dfrac{z_1}{z_2}\right)$ = Arg z_1 – Arg z_2 + $2k\pi$ ($k = 0$ or 1 or -1)
- Arg $\left(\dfrac{z}{\bar{z}}\right) = 2$ Arg z + $2k\pi$ ($k = 0$ or 1 or -1)
- Arg (z^n) = n Arg z + $2k\pi$ ($k = 0$ or 1 or -1)
- If Arg $\left(\dfrac{z_2}{z_1}\right) = \theta$, then Arg $\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.
- Arg $\bar{z}$ = – Arg z
- If arg $(z) = 0 \Rightarrow z$ is real.
  Note: Proper value of k must be chosen so, that R.H.S. of (i), (ii), (iii), (iv) lies in $(-\pi, \pi]$.
  All the above formulae are written on the basic of principal argument.

Illustration 8

Question: **Let z, z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an argand diagram by the points P, P_0, Q, A and the origin O respectively. Show that the triangles POP_0 and AOQ are congruent. Hence, or otherwise, prove that $|z \text{ J } z_0| = |z\bar{z}_0 \text{ J } 1|$.**

Solution: Given $OA = 1$ and $|z| = 1$

$\therefore$ $OP = |z - 0| = |z| = 1$

$\therefore$ $OP = OA$

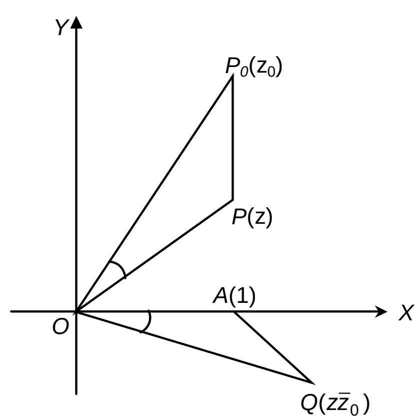
$OP_0 = |z_0 - 0| = |z_0|$

and $OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0|$

$= |z|\,|\bar{z}_0| = 1\,|z_0| = |z_0|$

$\therefore$ $OP_0 = OQ$

And $\angle P_0OP = \arg\left(\dfrac{z_0 - 0}{z - 0}\right) = \arg\left(\dfrac{z_0}{z}\right)$



$$= \arg\left(\frac{\bar{z}z_0}{z\bar{z}}\right) = \arg\left(\frac{\bar{z}z_0}{|z|^2}\right) = \arg\left(\frac{\bar{z}z_0}{1}\right) = -\arg\left(\overline{\bar{z}z_0}\right) = -\arg(z\bar{z}_0) = \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent.

Also, $PP_0 = AQ$

$\Rightarrow \quad |z - z_0| = |z\bar{z}_0 - 1|$

7. QUADRATIC EQUATIONS

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminants is non-negative i.e., ≥ 0.

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients *a, b, c* and $a \neq 0$.

Also, let us assume that the $b^2 - 4ac < 0$.

Now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

"A polynomial equation has at least one root."

As a consequences of this theorem, the following result, which is of immense importance, is arrived at :

"A polynomial equation of degree *n* has *n* roots."

Illustration 9

Question: **Solve** $x^2 + 2 = 0$.

Solution: We have $x^2 + 2 = 0$

or $x^2 = -2$ i.e., $x = \pm\sqrt{-2} = \pm\sqrt{2}\, i$

Illustration 10

Question: **Solve** $x^2 + x + 1 = 0$.

Solution: Here $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Illustration 11

Question: **Solve** $\sqrt{5}x^2 + x + \sqrt{5} = 0$.

Solution: Here, the discriminants of the equation is

$1^2 - 4 \times \sqrt{5} \times \sqrt{5} = 1 - 20 = -19$

Therefore, the solution are

$$\frac{-1 \pm \sqrt{-19}}{2\sqrt{5}} = \frac{-1 \pm \sqrt{19}\, i}{2\sqrt{5}}$$

PRACTICE PROBLEMS

PP1. If $(a + 2b) - i(2a - b) = 2i - 6$ then find a and b.

PP2. If $a = \frac{1+i}{\sqrt{2}}$ then prove that the value of a^{1929} is also equal to $\frac{1+i}{\sqrt{2}}$.

PP3. If $z_1 = 2 - 3i$ and $z_2 = 2 + 7i$ find $|z_1 - z_2|$ and $\arg(z_1 - z_2)$.

PP4. What is the polar form of $z = 1 - i\sqrt{3}$.

PP5. If $a + ib = \frac{(2+3i)^2}{2+i}$, find a and b.

PP6. Find the value of $i^{13} + i^{14} + i^{15} + i^{16}$.

PP7. If $X + iY = (x + iy)^{1/3}$ then prove that $4(X^2 - Y^2) = \frac{x}{X} + \frac{y}{Y}$.

PP8. If $|z_1| = |z_2| = |z_3| .... = |z_n| = 1$, then prove that

$$|z_1 + z_2 + z_3 + ... + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$$

PP9. Solve: $x^2 + 3 = 0$.

PP10. Solve: $x^2 + 3x + 9 = 0$.

PP11. Solve $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

If |*z* J 2 J 2*i*| = 1 then find the minimum value of |*z*|.

Solution:

Given $|\mathbf{z} - 2 - 2i| = 1$

Now $|2 + 2i| = |z - (z - 2 - 2i)|$

$\leq |z| + |z - 2 - 2i|$

$\Rightarrow \quad \sqrt{2^2 + 2^2} \leq |z| + 1 \quad \Rightarrow \quad |z| \geq 2\sqrt{2} - 1$

Hence minimum value of $|z|$ is $2\sqrt{2} - 1$.

Note that $z = \left(2 - \frac{1}{\sqrt{2}}\right) + \left(2 - \frac{1}{\sqrt{2}}\right)i$ satisfies $|z - (2 + 2i)| = 1$

and $\left|\left(2 - \frac{1}{\sqrt{2}}\right) + \left(2 - \frac{1}{\sqrt{2}}\right)i\right| (2\sqrt{2} - 1)$.

$2\sqrt{2} - 1$

***Example* 2:**

For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and |z_2 J 3 J 4*i* | = 5, find the minimum value of |z_1 J z_2|.

Solution:

First, we note that

$$|z_2| = |(z_2 - 3 - 4i) + (3 + 4i)| \leq |z_2 - 3 - 4i| + |3 + 4i| = 5 + 5 = 10$$

Hence $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

$= |12 - |z_2|| \geq |12 - 10| = 2 \qquad (\because |z_2| \leq 10 \;\therefore\; -|z_2| \geq -10)$

***Example* 3:**

Find the value of $(x^2 \text{ H } 5x)^2 \text{ H } x(x \text{ H } 5)$ for $x \text{ Z } \dfrac{\text{J } 5 \text{ H } i\sqrt{3}}{2}$

Solution:

$$x + 5 = \frac{-5 + i\sqrt{3}}{2} + 5 = \frac{5 + i\sqrt{3}}{2}$$

$$\therefore x(x + 5) = \left(\frac{-5 + i\sqrt{3}}{2}\right)\left(\frac{5 + i\sqrt{3}}{2}\right) = \frac{(-5)\,5 + 3i^2}{4}$$

$$= \frac{-25 - 3}{4} = -7$$

$\therefore$ The (required) value = $(-7)^2 - 7 = 49 - 7 = 42$

***Example* 4:**

Find two complex numbers satisfying the conditions that

(i) the sum of their real parts is 3

(ii) the product of their real parts is 2

(iii) their product is 5 – *i*

Solution:

Take the complex numbers as $a + ib$, $p + iq$

so that $a + p = 3; ap = 2 \Rightarrow \left.\begin{matrix} a = 2 \\ p = 1 \end{matrix}\right\}$ or $\left.\begin{matrix} a = 1 \\ p = 2 \end{matrix}\right\}$

Also $(a + ib)(p + iq) = ap - bq + i(bp + aq) = 5 - i$

Given $ap - bq = 5$; $aq + bp = -1$

Taking $a = 2$, $p = 1$; $bq = -3$ and $b + 2q = -1$

This gives $\left.\begin{matrix} b = -3 \\ q = 1 \end{matrix}\right\}$ or $\left.\begin{matrix} 2 + 2i \\ 1 - \frac{3}{2}i \end{matrix}\right\}$

The numbers are $\left.\begin{matrix} 2 - 3i \\ 1 + i \end{matrix}\right\}$ or $\left.\begin{matrix} 2 + 2i \\ 1 - \frac{3}{2}i \end{matrix}\right\}$

Thus there are two pairs of a complex numbers satisfying the requirements.

It may be verified that $a = 1$, $p = 2$, give the same set of numbers.

***Example* 5:**

Prove that

(i) $|Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = 2(|Z_1|^2 + |Z_2|^2)$

(ii) using above result, prove that $\left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$**, where** α**,** β **are complex numbers.**

Solution:

$$|Z_1 + Z_2|^2 = (Z_1 + Z_2)(\overline{Z}_1 + \overline{Z}_2) = Z_1\overline{Z}_1 + Z_2\overline{Z}_2 + Z_1\overline{Z}_2 + Z_2\overline{Z}_1$$

$$|Z_1 - Z_2|^2 = (Z_1 - Z_2)(\overline{Z}_1 - \overline{Z}_2) = Z_1\overline{Z}_1 + Z_2\overline{Z}_2 - Z_1\overline{Z}_2 - Z_2\overline{Z}_1$$

Adding, $|Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = 2(Z_1\overline{Z}_1 + Z_2\overline{Z}_2)$

$$= 2\,(|Z_1|^2 + |Z_2|^2)$$

Now, for the second part,

$$\left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha + \sqrt{\alpha^2 - \beta^2}\right|$$

$$= \frac{1}{2}\left\{\left|2\alpha - 2\sqrt{\alpha^2 - \beta^2}\right| + \left|2\alpha + 2\sqrt{\alpha^2 - \beta^2}\right|\right\}$$

$$= \frac{1}{2}\left\{\left|\alpha + \beta + \alpha - \beta - 2\sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha + \beta + \alpha - \beta + 2\sqrt{\alpha^2 - \beta^2}\right|\right\}$$

$$= \frac{1}{2}\left\{\left|\sqrt{\alpha + \beta} - \sqrt{\alpha - \beta}\right|^2 + \left|\sqrt{\alpha + \beta} + \sqrt{\alpha - \beta}\right|^2\right\}$$

$$= \frac{1}{2}\left\{|Z_1 - Z_2|^2 + |Z_1 + Z_2|^2\right\}$$

$$= \frac{1}{2}\left\{2(|Z_1|^2 + |Z_2|^2)\right\}$$

$$= \left|\sqrt{\alpha + \beta}\right|^2 + \left|\sqrt{\alpha - \beta}\right|^2$$

$$= |\alpha + \beta| + |\alpha - \beta|$$

***Example* 6:**

For every real $b \geq 0$, find all the complex numbers Z satisfying $2|Z| - 4bZ + 1 + ib = 0$.

Solution:

Let $Z = x + iy$. The equation is

$$2\sqrt{x^2 + y^2} - 4b(x + iy) + 1 + ib = 0$$

Real part: $2\sqrt{x^2 + y^2} - 4bx + 1 = 0$ … (i)

Imaginary Part: $-4by + b = 0$ …(ii)

From (ii) either $b = 0$ and in that case from (i), $2\sqrt{x^2 + y^2} + 1 = 0$ and this equation is not satisfied for any (x, y)

$\therefore$ $b = 0$, there is no solution for the equation, If $b \neq 0$ but > 0; $-4y + 1 = 0$ from (ii)

i.e. $y = \frac{1}{4}$

Substituting $y = \frac{1}{4}$ in (i)

$$2\sqrt{x^2 + \frac{1}{16}} = 4bx - 1 \quad \text{… (iii)}$$

This requires that $4bx - 1 > 0$ i.e. $x > \frac{1}{4b}$ and $b > 0$ and hence $x > 0$

Squaring (iii)

$$4\left(x^2 + \frac{1}{16}\right) = 16b^2x^2 - 8bx + 1$$

$$x^2(16b^2 - 4) - 8bx + \frac{3}{4} = 0$$

Roots are $\frac{8b \pm \sqrt{16b^2 + 12}}{2(16^2 - 4)} = \frac{4b \pm \sqrt{4b^2 + 3}}{16b^2 - 4}$

If $16b^2 - 4 < 0$, which in effect means that $b < \frac{1}{2}$ (note already $b > 0$), the values of x are such that

(i) for the + sign $x < 0$ while the requirement is $x > 0$

(ii) for the – sign, even if $x > 0$, the condition $x > \frac{1}{4b}$ is not satisfied.

$\therefore$ For $0 < b < \frac{1}{2}$, there is no solution.

For $b > \frac{1}{2}$, the solution is

$$Z = \frac{4b + \sqrt{4b^2 + 3}}{16b^2 - 4} + \frac{i}{4}$$

Example 7:

For complex numbers *z* and *w*, prove that $|z|^2 w - |w|^2 z = z - w$ **if and only if** $z = w$ **or** $z\bar{w} = 1$**.**

Solution:

$\dfrac{z}{w} = \dfrac{|z|^2 + 1}{|w|^2 + 1}$ = purely real number

$\Rightarrow$ $\dfrac{z}{w}$ is purely real i.e., $\dfrac{z}{w} = \overline{\left(\dfrac{z}{w}\right)}$ $\Rightarrow$ $z\bar{w} = \bar{z}w$ …(i)

$|z|^2 w - |w|^2 z = z - w$

$z\bar{z}w - w\bar{w}z = z - w$

from (i), $z\bar{w}(z - w) = z - w$ …(ii)

$(z\bar{w} - 1)(z - w) = 0$ $\Rightarrow$ $z = w$ or $z\bar{w} = 1$

Conversely if $z = w$, then LHS = RHS = 0

$z\bar{w} = 1$ $\Rightarrow$ then from (i) and (ii) L.H.S. = R.H.S. = $z - w$

Example 8:

Find all non-zero complex numbers satisfying $\bar{Z}$ Z iZ^2.

Solution:

Let $Z = x + iy$; $\bar{Z} = x - iy$; $Z^2 = x^2 - y^2 + 2ixy$

$\therefore$ The equation is $x - iy = i(x^2 - y^2 + 2ixy)$

$\therefore$ Equating real and imaginary parts

$x = -2xy$ … (i)

$-y = x^2 - y^2$ …(ii)

(i) gives either $x = 0$, in that case $y = 0$; $y = 1$

or $y = -\dfrac{1}{2}$, in that case $\dfrac{1}{4} + \dfrac{1}{2} = x^2$

$\therefore$ $x = \pm\dfrac{\sqrt{3}}{2}$

$\therefore$ The non-zero Z, satisfying the equation are

$$Z_1 = i\ ;\ Z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i\ ;\ Z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Example* 9*:

Solve the equation; *Z* H *a* | *Z* H 1| H *i* Z 0 (*a* is a real number ≥ 1).

Solution:

Taking $Z = x + iy$, the equation reduces to $x + iy + a\sqrt{x^2 + 2x + 1 + y^2} + i = 0$.

Imaginary $= 0 \Rightarrow y = -1$

Real part $= 0 \Rightarrow x + a\sqrt{x^2 + 2x + 1 + y^2} = 0$

Eliminating y, the equation in x is $x^2 = a^2 (x^2 + 2x + 2)$

$x^2 (a^2 - 1) + 2a^2x + 2a^2 = 0$

This gives real x only if $4a^4 - 8a^2 (a^2 - 1) \geq 0$

i.e if $-a^4 + 2a^2 \geq 0$ i.e., if $0 \leq a^2 \leq 2$

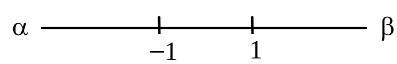
and $a \geq 1$, the value of a are $1 \leq a \leq \sqrt{2}$

$x = \dfrac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1}, < 0$ for the negative sign and for the positive sign also $x < 0$.

Hence the solutions are $Z = \left\{\dfrac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1}\right\} - i$ where $1 \leq a \leq \sqrt{2}$.

Example* 10*:

Let *a*, *b*, *c*, be real. If $ax^2 + bx + c = 0$ has two real roots ~, where ~ < –1 and > 1, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.



Solution:

Since the roots are real and different $b^2 - 4ac > 0$

$\alpha < -1, \beta > -1 \Rightarrow \alpha + 1 < 0$ and $\beta + 1 > 0$ … (i)

$\alpha < 1, \beta > 1 \Rightarrow \alpha - 1 < 0$ and $\beta - 1 > 0$ … (ii)

From (i), $(\alpha + 1)(\beta + 1) < 0$ i.e., $\alpha\beta + (\alpha + \beta) + 1 < 0$

$\frac{c}{a} - \frac{b}{a} + 1 < 0$ … (iii)

From (ii), $(\alpha - 1)(\beta - 1) < 0$

$\alpha\beta - (\alpha + \beta) + 1 < 0$

$\frac{c}{a} + \frac{b}{a} + 1 < 0$ … (iv)

Combining (iii) and (iv), we get $\frac{c}{a} + \left|\frac{b}{a}\right| + 1 < 0$

EXERCISE – I

1. Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$.

2. If $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$, where α and β are complex cube roots of unity, show that $xyz = a^3 + b^3$.

3. Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{17}+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^{17}=-1$.

4. Graph the complex numbers on the Argand plane $3 + 2i$, $-4 - 5i$, -5, $1 - 2i$.

5. Express the numbers in the form $r(\cos\theta + i\sin\theta)$:

(i) $1 + i\tan\alpha$ (ii) $1 - \sin\alpha + i\cos\alpha$.

6. Find the square root of following:

(i) $7 - 24i$ (ii) $-5+12i$

(iii) $4ab-2\left(a^2-b^2\right)i$ (iv) $x^2+\dfrac{1}{x^2}+4i\left(x-\dfrac{1}{x}\right)-6$

7. Find x and y in the following equations:

(i) $(x+iy)(2-3i)=4+i$ (ii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

8. Express the following in the form of $a+ib$

(i) $\dfrac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i}$ (ii) $(1-i)^4$

9. For any two complex numbers z_1 and z_2, prove that $\text{Re}(z_1z_2)=\text{Re}\,z_1\,\text{Re}\,z_2-\text{Im}\,z_1\,\text{Im}\,z_2$.

10. If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$, then find value of $\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right)$.

11. Find the value of $\sqrt[4]{-64}$.

12. Find the real θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real.

13. Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

14. Find modulus of the following complex numbers:

(i) $\frac{1+2i}{1-3i}$ (ii) $\frac{1+i}{1-i}-\frac{1-i}{1+i}$

15. Find argument of following complex numbers:

(i) $\frac{1+i}{1-i}$ (ii) $-\sqrt{3}-i$

16. Convert the following complex numbers in polar form:

(i) $\frac{1+2i}{1-3i}$ (ii) $1+i$

17. If $|2z-1|=|z-2|$, then find value of $|z|$.

18. If $x=\sqrt{2}\,i-1$ find the value of $x^4+4x^3+6x^2+4x+9$.

19. Express $(1+a^2)(1+b^2)$ as sum of two squares.

20. Find the value of $\frac{(1+i)^{4n+5}}{(1-i)^{4n+3}}$.

EXERCISE – II

1. Find out the smallest positive integer for which $(1+i)^{2n} = (1-i)^{2n}$.

2. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find out $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

3. For a complex number *z*, then find the minimum value of $|z| + |z-2|$.

4. Let $z = 1 - t + i\sqrt{(t^2 + t + 2)}$, where *t* is a real parameter. Find out the locus of *z* in the Argand plane.

5. If $|z_1 + z_2| = |z_1| + |z_2|$, then find out $\arg z_1 - \arg z_2$.

6. If $z_1 = \frac{-1+3i}{2}$ and $z_2 = \frac{-1-3i}{2}$, then find out the value of $z_1^3 + z_2^3 - 3z_1z_2$.

7. If one root of the equation $ix^2 - 2(1+i)x + (2-i) = 0$ is $2-i$, then find out other root.

8. Let *z* and *w* be two complex numbers such that $|z| \le 1, |w| \le 1$ and $|z + iw| = |z - i\overline{w}| = 2$, then find out *z.*

9. If $\arg z < 0$, then find out $\arg(-z) - \arg(z)$.

10. If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.

11. Find out the complex number *z* satisfying $|z + \bar{z}| + |z - \bar{z}| = 2$ and $|i\,z - 1| + |z - i| = 2$.

12. If *z* satisfies $iz^2 = \bar{z}^2 + z$, then find out $\arg(z)$.

13. If $z = \frac{1+3i}{1+i}$, then prove that : (i) $\text{Re}(z) = 2\text{Im}(z)$ (ii) $|z| = 5$

14. If $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other then find out (x, y).

15. Solve: $x^{-2} + 2x^{-1} - 3 = 0$.

16. Solve for *x* : $4^x + 9^x = 2\,(6^x)$.

17. Find the real roots of the equation $x^2 + 5|x| + 4 = 0$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. $a = -2, b = -2.$

PP3. $|z_1 - z_2| = 10$ and $\text{Arg}(z_1 - z_2) = -\pi/2$

PP4. $z = 2e^{i(-\pi/3)}$

PP5. $a = \frac{2}{5}, b = \frac{29}{5}$

PP6. 0

PP9. $\pm\sqrt{3}\ i$

PP10. $\frac{-3 \pm 3\sqrt{3}\,i}{2}$

PP11. $\frac{\sqrt{2} \pm \sqrt{34}\,i}{2\sqrt{3}}$

EXERCISE – I

1. -1

5. (i) $\sec\alpha\,(\cos\alpha + i\sin\alpha)$

(ii) $r(\cos\theta + i\sin\theta)$, $r = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$, $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$.

6. (i) $\pm(4-3i)$ (ii) $\pm(2+3i)$ (iii) $\pm[(a+b) - i(a-b)]$

(iv) $\pm\left(x - \frac{1}{x} + 2i\right)$

7. (i) $x = \frac{5}{13}, y = \frac{14}{13}$ (ii) $x = 3, y = -1$

8. (i) $1 + 2\sqrt{2}\,i$ (ii) -4

10. $A^2 + B^2$

11. $\pm 2(1 \pm i)$

12. $\theta = n\pi$

13. $\frac{63}{25} + \frac{16}{25}i$

14. (i) $\frac{1}{\sqrt{2}}$ (ii) 2

15. (i) $\frac{\pi}{2}$ (ii) $-\frac{5\pi}{6}$

16. (i) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (ii) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

17. 1

18. 12

19. $(1-ab)^2 + (a+b)^2$

20. 2

EXERCISE – II

1. 2

2. 0

3. 2

4. a hyperbola

5. π

6. –1

7. $-i$

8. 1 or –1

9. π

10. 2

11. $i, \ -i, \ \frac{1}{i}, \ \frac{1}{i^3}$

12. $\frac{3\pi}{4}, \ -\frac{\pi}{4}$

14. $(1, -4), \ (-1, -4)$

15. $x = \frac{-1}{3}, 1$

16. $x = 0$

17. no real root