LESSON 4

MATHEMATICAL INDUCTION

1. INTRODUCTION

The word induction means the method of reasoning about a general statement from the conclusion of particular cases. Inductions starts with observations. It may be true but then it must be so proved by the process of reasoning. Else it may be false but then it must be shown by finding a counter example where the conjecture fails.

In mathematics there are some results or statements that are formulated in terms of *n*, where $n \in N$. To prove such statements we use a well suited method, based on the specific technique, which in known as **principle of mathematical induction**.

2. PROPOSITION

A statement which is either true or false is called a proposition or statement.

$P(n)$ denotes a proposition whose truth value depends on natural variable '*n*'.

For example $1^2 + 2^2 + 3^2 + ...n^2 = \frac{n(n+1)(2n+1)}{6}$ is a proposition whose truth value depends on natural number *n*.

We write $P(n): 1^2 + 2^2 + 3^2 + ...n^2 = \frac{n(n+1)(2n+1)}{6}$,

where *P*(4) means $1^2 + 2^2 + 3^2 + 4^2 = \frac{4(4+1)(8+1)}{6}$

To prove the truth of proposition $P(n)$ depending on natural variable *n*, we use mathematical induction.

3. FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

The statement $P(n)$ is true for all $n \in N$ if

(i) $P(1)$ is true.

(ii) $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

The above statement can be generalized as $P(n)$ is true for all $n \in N$ and $n \geq k$ if

(i) $P(k)$ is true.

(ii) $P(m)$ is true $(m > k)$ $\Rightarrow$ $P(m+1)$ is true.

4. APPLICATION OF FIRST PRINCIPLE OF MATHEMATICAL INDUCTION (Working Rule)

To prove any statement $P(n)$ to be true for all $n \geq k$ with the help of first principle of mathematical induction we follow the following procedure:

Step (i) Check if the statement is true or false for $n = k$.

Step (ii) Assume the statement is true for $n = m$.

Step (iii) Prove the statement is true for $n = m + 1$.

Illustration 1

Question: **Prove by the principle of mathematical induction that for all $n \in N$:**

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Solution: Let $P(n)$ be the statement given by

$$P(n):\ \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Step I: We have, $P(1):\ \frac{1}{1.2}=\frac{1}{1+1}$

Since, $\frac{1}{1.2}=\frac{1}{1+1}=\frac{1}{2}$

So, $P(1)$ is true.

Step II: Let $P(m)$ be true, then $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{m(m+1)}=\frac{m}{m+1}$ (i)

We shall now show that $P(m+1)$ is true. If $P(m)$ is true.

For this we have to show that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......+\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+1+1)}=\frac{m+1}{(m+1)+1}$$

Now, $$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......\frac{1}{m(m+1)}+\frac{1}{(m+1)\,(m+1+1)}$$

$$=\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+......\frac{1}{m(m+1)}\right]+\frac{1}{(m+1)\,(m+1+1)}$$

$$=\frac{m}{m+1}+\frac{1}{(m+1)((m+1)+1)} \qquad \text{[using (i)]}$$

$$=\frac{1}{(m+1)}\left\{\frac{m}{1}+\frac{1}{m+2}\right\}=\frac{1}{(m+1)}\frac{(m^2+2m+1)}{(m+2)}=\frac{(m+1)^2}{(m+1)(m+2)}=\frac{m+1}{m+2}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

Illustration 2

Question: **For every positive integer *n*, prove that 7^n J 3^n is divisible by 4.**

Solution: We have $P(n): 7^n - 3^n$ is divisible by 4

We note that $P(1): 7^1 - 3^1 = 4$, which is divisible by 4. Thus $P(n)$ is true for $n = 1$

Let $P(k)$ be true for some natural number *k*.

i.e., $P(k): 7^k - 3^k$ is divisible by 4.

We get $7^k - 3^k = 4d$, where $d \in N$ …(i)

Now, we wish to prove that $P(k+1)$ is true whenever $P(k)$ is true.

i.e., we have show $7^{k+1} - 3^{k+1} = 4m$

Now, $7^{(k+1)} - 3^{(k+1)} = 7^{(k+1)} - 7.3^k + 7\ 3^k - 3^{(k+1)}$

$$= 7(7^k - 3^k) + (7-3)3^k = 7(4d) + (7-3)3^k$$

$$= 7(4d) + 4.3^k \qquad \text{[using (i)]}$$

$$= 4(7d + 3^k) = 4m$$

From the last line, we see that $7^{(k+1)} - 3^{(k+1)}$ is divisible by 4.

Thus, $P(k+1)$ is true when $P(k)$ is true.

Therefore, by principle of mathematical induction the statement is true for every positive integer *n*.

Illustration 3

Question: **Prove that $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in N$.**

Solution: Let the statement $P(n)$ be defined as

$P(n): 2.7^n + 3.5^n - 5$ is divisible by 24.

Now, $P(n)$ is true for $n = 1$, since $2.7 + 3.5 - 5 = 24$, which is divisible by 24.

Assume that $P(k)$ is true. i.e. $2.7^k + 3.5^k - 5 = 24q$, when $q \in N$ …(i)

Now, we wish to prove that $P(k+1)$ is true whenever $P(k)$ is true.

We have, $2.7^{k+1} + 3.5^{k+1} - 5 = 2.7^k.7^1 + 3.5^k.5^1 - 5$

$= 7[2.7^k + 3.5^k - 5 - 3.5^k + 5] + 3.5^k.5 - 5$

$= 7[24q - 3.5^k + 5] + 15.5^k - 5$ [using (i)]

$= 7 \times 24q - 21.5^k + 35 + 15.5^k - 5$

$= 7 \times 24q - 6.5^k + 30$

$= 7 \times 24q - 30\left(5^{k-1} - 1\right) = 7 \times 24q - 30\,(4p)$

[since $\left(5^{k-1} - 1\right)$ is a multiple of 4 as $x^n - y^n$ is divisible by $x = y$]

$= 7 \times 24q - 120p = 24(7q - 5p)$

$= 24 \times r$, $r = 7q - p$, is some natural number …(ii)

The expression on the R.H.S. of (i) is divisible by 24.

Thus $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for all $n \in N$

Illustration 4

Question: **Prove the rule of exponents $(ab)^n = a^n b^n$ by using principle of mathematical induction for every natural number.**

Solution: Let $P(n)$ be the given statement i.e, $P(n): (ab)^n = a^n b^n$

We note that $P(n)$ is true for $n = 1$ since $(ab)^1 = a^1 b^1$

Let $P(k)$ be true, i.e., $(ab)^k = a^k b^k$ …(i)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Now, we have $(ab)^{k+1} = (ab)^k (ab)$

$$= \left(a^k\, b^k\right)(ab) \qquad \text{[by (i)]}$$

$$= \left(a^k . a^1\right)\left(b^k . b^1\right) = a^{k+1} . b^{k+1}$$

Therefore, $P(k+1)$ is also true whenever $P(k)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Illustration 5

Question: **Using mathematical induction, show that**

$$\cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos\left(2^{n-1}\theta\right) = \frac{\sin 2^n\theta}{2^n \sin\theta}, \ \forall\ n \in N .$$

Solution: Let $P(n): \cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos\left(2^{n-1}\theta\right) = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

Step I: For $n = 1$

L.H.S. of (i) = $\cos\theta$ and R.H.S. of (i) = $\dfrac{\sin 2\theta}{2\sin\theta} = \cos\theta$

Therefore, $P(1)$ is true.

Step II: Assume it is true for $n = k$, then

$$p(k): \cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos\left(2^{k-1}\theta\right) = \frac{\sin 2^k\theta}{2^k \sin\theta} \qquad \ldots\text{(i)}$$

Step III: For $n = k+1$

$$P(k+1): \cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos\left(2^{k-1}\theta\right)\cos\left(2^k\theta\right) = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta}$$

L.H.S. = $\cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos\left(2^{k-1}\theta\right)\cos\left(2^k\theta\right)$

$$= \frac{\sin\left(2^k\theta\right)}{2^k \sin\theta} . \cos\left(2^k\theta\right) = \frac{2\sin\left(2^k\theta\right)\cos\left(2^k\theta\right)}{2^{k+1}\sin\theta} \qquad \text{[using (i)]}$$

$$= \frac{\sin\left(2 . 2^k\theta\right)}{2^{k+1}\sin\theta} = \frac{\sin\left(2^{k+1}\theta\right)}{2^{k+1}\sin\theta} = \text{R.H.S.}$$

This shows that the $P(k+1)$ is true if $P(k)$ is true.

Hence by the principle of mathematical induction, the result is true for all $n \in N$.

Illustration 6

Question: **Prove by induction that the sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for all $n \in N$.**

Solution: Let $P(n)$ be the statement given by

$P(n): S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3

Step I: We have, $P(1): S_1 = 1^3 + 3(1)^2 + 5(1) + 3$ is divisible by 3

Since $1^3+3(1)^2+5(1)+3=12$, which is divisible by 3

$\therefore$ $P(1)$ is true.

Step II: Let $P(m)$ be true. Then

$S_m = m^3+3m^2+5m+3$ is divisible by 3

$\Rightarrow$ $S_m = m^3+3m^2+5m+3 = 3\lambda$, for some $\lambda \in N$ …(i)

We now wish to show that $P(m+1)$ is true. For this we have to show that

$(m+1)^3+3(m+1)^2+5(m+1)+3$ is divisible by 3

Now, $(m+1)^3+3(m+1)^2+5(m+1)+3 = (m^3+3m^2+5m+3)+3m^2+9m+9$

$= 3\lambda+3(m^2+3m+3) = 3[\lambda+m^2+3m+3]$ [using (i)]

$= 3\mu$, where $\mu = \lambda+m^2+3m+3 \in N$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by the principle of mathematical induction the statement is true for all $n \in N$.

Illustration 7

Question: **Show by using principle of mathematical induction that**

$$1.3+2.3^2+3.3^3+......+n.3^n = \frac{(2n-1)3^{n+1}+3}{4}.$$

Solution: Let $P(n): 1.3+2.3^2+3.3^3+......+n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

Step I: When n = 1, L.H.S. =1.3 = 3

and R.H.S. = $\frac{(2n-1)3^{n+1}+3}{4} = \frac{(2.1-1)3^2+3}{4} = \frac{12}{4} = 3$

Hence $P(1)$ is true.

Let $P(m)$ be true

$\Rightarrow$ $1.3+2.3^2+3.3^3+......+m.3^m = \frac{(2m-1)3^{m+1}+3}{4}$ ….(i)

To prove $P(m+1)$ is true i.e.,

$$1.3+2.3^2+......+m.3^m+(m+1).3^{m+1} = \frac{(2m+1)3^{m+2}+3}{4}$$

Adding $(m+1).3^{m+1}$ to both sides of (i), we get

$1.3+2.3^2+......+m.3^m+(m+1).3^{m+1}$

$$= \frac{(2m-1)3^{m+1}+3}{4} + (m+1).\,3^{m+1}$$

$$= \frac{\{2m-1+4(m+1)\}.\,3^{m+1}+3}{4} = \frac{(2m+1)\;3^{m+2}+3}{4}$$

Hence $P(m+1)$ is true whenever $P(m)$ is true.

It follows that $P(n)$ is true for all natural numbers *n*.

Illustration 8

Question: **Using the principle of mathematical induction, show that $11^{n+2}+12^{2n+1}$, where *n* is a natural number, is divisible by 133.**

Solution: Let $f(n) = 11^{n+2}+12^{2n+1}$

Let $P(n) : f(n)$ i.e. $11^{n+2}+12^{2n+1}$ is divisible by 133

Now, $f(1) = 11^3 + 12^3 = (11+12)(121-11\times 12+144)$

$= 23 \times 133$, where is divisible by 133

$\therefore$ $P(1)$ is true. …(A)

Let $P(m)$ be true $\Rightarrow$ $f(m) = 11^{m+2}+12^{2m+1}$ is divisible by 133

$\Rightarrow$ $f(m) = 11^{m+2}+12^{2m+1} = 133k$, where *k* is an integer …(i)

Now, $f(m+1) = 11^{m+3}+12^{2m+3}$

$= 11.11^{m+2}+12^{2m+1}.12^2 = 11(11)^{m+2}+12^{2m+1}.144$

Now we divide $f(m+1)$ by $f(m)$

$$11^{m+2}+12^{2m+1}\Big)\, 11.(11)^{m+2}+144.12^{2m+1}\Big(11$$

$$11.(11)^{m+2}+11.12^{2m+1}$$

$$- \qquad\qquad -$$

$$\overline{\qquad\qquad\qquad 133.12^{2m+1}}$$

$\therefore$ $f(m+1) = 11.f(m)+133.12^{2m+1} = 11\times 133k + 133.12^{2m+1}$ [from (i)]

$= 133\left(11k + 12^{2m+1}\right)$, which is divisible by 133

$\therefore$ $P(m+1)$ is true whenever $P(m)$ is true …(B)

From (A) and (B) it follows that $P(n)$ is true for every natural number *n* i.e. $11^{n+2}+12^{2n+1}$ is divisible by 133 for every natural number *n*.

PRACTICE PROBLEMS

PP1. Prove by the principle of mathematical induction for all $n \in N$, $1^3 + 2^3 + ...n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

PP2. Prove by the principle of mathematical induction for all $n \in N$,
$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)......\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}$.

PP3. Prove by mathematical induction that 3^{2n} when divided by 8 the remainder is always 1, $n \in N$.

PP4. Prove that $1^2 + 2^2 + ... + n^2 > \frac{n^3}{3}$, $n \in N$.

PP5. Prove that $2^n > n$ for all positive integers n.

PP6. Prove by mathematical induction $3^{2n+2} - 8n - 9$ is divisible by 64.

PP7. Prove by mathematical induction $3.6 + 6.9 + 9.12 + ... + (3n)(3n+3) = 3n(n+1)(n+2)$.

PP8. Prove by mathematical induction for $\forall n \in N$,
$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\theta/2}$.

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Let $p(m)$ be the statement "$3^m > m$". If $P(m)$ is true, prove that $P(m+1)$ is true.

Solution:

$P(n): 3^m > m$

We wish to prove that $P(m+1)$ is true.

i.e., $3^{m+1} > m+1$

$P(m)$ is true.

$\Rightarrow \quad 3^m > m$

$\Rightarrow \quad 3.\ 3^m > 3.m$

$\Rightarrow \quad 3^{m+1} > m + 2m$

$\Rightarrow \quad 3^{m+1} > m+1$ as $2m > 1$ for $\forall\, m \in N$

$\Rightarrow \quad P(m+1)$ is true.

***Example* 2:**

If $P(m)$ is the statement "$m(m+1)(m+2)$ is divisible by 12". Prove that the statements $P(3)$ and $P(4)$ are true but that $P(5)$ is not true.

Solution:

$P(m): m(m+1)(m+2)$

$P(3): 3.4.\ 5 = 60$ is divisible by 12

$P(4): 4.5.6 = 120$ is divisible by 12

$P(5): 5.6.7 = 210$

Clearly it is not true.

***Example* 3:**

If $P(n)$ is the statement "$2^n \geq 3n$ and if $P(r)$ is true, prove that $P(r+1)$ is true.

Solution:

$P(r)$ is true.

$\Rightarrow \quad 2^r \geq 3r$

$\Rightarrow \quad 2.2^r \geq 6r$

$\Rightarrow \quad 2^{r+1} \geq 3r+3 \qquad \because \; 3r \geq 3 \;\Rightarrow\; 3r+3r \geq 3r+3$

$\Rightarrow \quad 2^{r+1} \geq 3(r+1)$

$\Rightarrow \quad P(r+1)$ is true.

***Example* 4:**

Prove that principle of mathematical induction $41^n - 14^n$ is a multiple of 27.

Solution:

Let $P(n): 41^n - 14^n$ is a multiple of 27

Step 1: $P(1): 41^1 - 14^1$ is a multiple of 27

We have $41^1 - 14^1 = 27$, which is a multiple of 27

$\Rightarrow \; P(1)$ is true.

Step 2: Let $P(n)$ be true, then

$41^n - 14^n$ is a multiple of 27

$\Rightarrow \; 41^n - 14^n = 27\lambda$ for $\forall \lambda \in N$

Now, $41^{n+1} + 14^{n+1} = 41^{n+1} - 41 \times 14^n + 41 \times 14^n - 14^{n+1}$

$= 41(41^n - 14^n) + 14^n(41 - 14)$

$= 41 \times 27\lambda + 14^n 27$

$= 27[41\lambda + 14^n]$, which is multiple of 27.

$\therefore \quad P(n+1)$ is true.

***Example* 5:**

Using principle of mathematical induction, prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all $n \in N$.

Solution:

Let $P(n): x^{2n} - y^{2n}$ is divisible by $(x+y)$

Step 1: $P(1): x^2 - y^2$ is divisible by $x+y$

$\Rightarrow$ $P(1)$ is true.

Step 2: Let $P(n)$ be true, then

$x^{2n} - y^{2n}$ is divisible by $x+y$

$\Rightarrow$ $x^{2n} - y^{2n} = \lambda(x+y)$

We wish to prove that $P(n+1)$ is true.

$P(n+1): x^{2m+2} - y^{2m+2}$

$= x^{2m+2} - x^{2m}y^2 + x^{2m}y^2 - y^{2m+2}$

$= x^{2m}[x^2 - y^2] + y^2[x^2 - y^{2m}]$

$= x^{2m}[x^2 - y^2] + y^2\lambda(x+y)$

$= (x+y)[x^{2m}(x-y) + y^2\lambda]$

$\Rightarrow$ $P(n+1)$ is divisible by $x+y$.

$\Rightarrow$ $P(n+1)$ is true.

***Example* 6:**

By mathematical induction, prove that $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) = n+1$ for $n \in N$.

Solution:

Let $P(n): \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).......\left(1+\frac{1}{n}\right) = n+1$

Step 1: $P(1): \left(1+\frac{1}{1}\right) = 1+1$

$2 = 2$

$\therefore$ $P(1)$ is true.

Step 2: Let $P(m)$ be true, then

$$P(m)=\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)......\left(1+\frac{1}{m}\right)=m+1 \qquad ...(i)$$

Now $P(m)$ is true.

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).....\left(1+\frac{1}{m}\right)=m+1 \qquad \text{using (i)}$$

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).....\left(1+\frac{1}{m}\right)\left(1+\frac{1}{m+1}\right)=(m+1)\left(1+\frac{1}{m+1}\right)$$

[multiplying both side by $\left(1+\frac{1}{m+1}\right)$]

$$=\frac{(m+1)(m+2)}{mn}$$

$$=m+2$$

$\therefore \quad P(m+1)$ is true.

$\Rightarrow \quad P(m)$ is true.

$\Rightarrow \quad P(m+2)$ is true.

***Example* 7:**

Prove by principle of induction that $4+8+12+......+4n=2n(n+1)$ for $\forall n \in N$.

Solution:

Let $P(n)=4+8+12+......+4n=2n(n+1)$

Step 1: $P(1)$ = 4 = 2. 1(1+1) = 4

$\therefore \ P(1)$ is true.

Step 2: Let $P(m)$ be true, then

$$P(m)=4+8+12+......+4m=2m(m+1) \qquad ...(i)$$

We wish to prove that $P(m+1)$ is true.

$$4+8+12+.....+4(m+1)=2(m+1)(m+1+1)$$

Now $4+8+12+......+4m+4(m+1)$

$$=4(1+2+3+.....+m)+4(m+1)$$

$$=\frac{4m(m+1)}{2}+4(m+1)$$

$$=2m(m+1)+4(m+1)$$

$$=2(m+1)(m+2)$$

$\Rightarrow \quad P(m+1)$ is true.

***Example* 8:**

Prove by mathematical induction $(n+3)^2 \le 2^{n+3}$

Solution:

$P(n):(n+3)^2 \le 2^{n+3}$

Step 1: $P(1):(1+3)^2 \le 2^{1+3}$

$16 \le 2^4$

$\therefore \quad P(1)$ is true.

Step 2: Let $P(m)$ be true, then

$P(m):(m+3)^2 \le 2^{m+3}$

$\Rightarrow \quad (m+3)^2 + (2m+7) \le 2^{m+3} + (2m+7)$

$\Rightarrow \quad (m+4)^2 \le 2^{m+3} + (m+3)^2 \qquad \left\{\begin{array}{l} \because \; 2m+7 < (m+3)^3 \\ \therefore \; 2^{m+3} + (2m+7) < 2^{m+3} + (m+3)^2 \end{array}\right\}$

$\Rightarrow \quad (m+4)^2 \le 2^{m+3} + 2^{m+3} \qquad \left\{\begin{array}{l} \because \; (m+3)^2 \le 2^{m+3} \\ \Rightarrow \; (m+3)^2 + 2^{m+3} \le 2^{m+3} + 2^{m+3} \end{array}\right\}$

$\Rightarrow \quad (m+4)^2 \le 2.2^{m+3}$

$\Rightarrow \quad (m+4)^2 \le 2^{m+4}$

$\Rightarrow \quad ((m+1)+3)^2 \le 2^{(m+1)+3}$

$\Rightarrow \quad P(m+1)$ is true.

EXERCISE - I

1. Find the smallest positive integer 'n' for which $2^n(1\times 2\times 3\times .....\times n) < n^n$ holds.

2. Find the smallest positive integer for which $(1\times 2\times 3\times ....\times n) < \left(\frac{n+1}{2}\right)^n$ holds.

3. Let $P(n)$: $n^2 - n + 41$ is a prime number, then prove that $P(41)$ is not true.

4. Consider each of the following statements and state with reasons if they are correct or incorrect. Which of the following statement is false?

(i) $P(n)$: $X^{2n} - Y^{2n}$ is divisible by $X + Y$

(ii) $P(n)$: $3^{2n+2} - 8n - 9$ is divisible by 8

(iii) $P(n)$: $n(n+1)\ (n+5)$ is multiple of 3

(iv) $P(n)$: $1 + 2 + 3 + ... + n > \frac{1}{8}(2n+1)^2$

5. Let $P(n)$: $(1+x)^n - 1 - nx$, $n > 1,\ x > -1,\ x \neq 0$ then prove that

(i) $P(n)$ is divisible by x

(ii) $P(n)$ is divisible by x^2

(iii) $P(n)$ is not a multiple of x^3

6. Prove by the principle of mathematical induction:

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)} \text{ for all } n \in N .$$

7. Prove by the principle of mathematical induction that
$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all $n \in N$.

8. Prove by the principle of mathematical induction that $3^{2n+2} - 8n - 9$ for all $n \in N$ is divisible by 8.

9. Prove by mathematical induction $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ for all $n \in N$.

10. Prove by using induction $n(n+1)(n+5)$ is a multiple of 3.

EXERCISE - II

1. Prove by the principle of mathematical induction:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+......+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)} \text{ for all } n\in N.$$

2. Prove by the principle of mathematical induction that $a+ar+ar^2+......+ar^{n-1}=\frac{a(r^n-1)}{r-1}$ for all $n\in N$.

3. Prove by the principle of mathematical induction

$1(1!)+2(2!)+3(3!)+......+n(n!)=(n+1)!-1$ for $\forall n\in N$

4. Prove by mathematical induction $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number for $n\in N$.

5. Prove by using induction that $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+......+\frac{1}{(1+2+3+......n)}=\frac{2n}{(n+1)}$ for all $n\in N$.

6. Prove by mathematical induction, that $(r+1)(r+2)(r+3)(r+4)(r+5)$ in divisible by 120 $\forall\ n\in N$.

7. Prove by using induction that $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+......+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$ for all $n\in N$.

8. Prove by using induction 41^n-14^n is a multiple of 27.

9. Prove by using mathematical induction that $(2n+7)<(n+3)^2$ for all $n\in N$.

10. Show by using mathematical induction that $15^{2n-1}+1$ is a multiple of 16.

ANSWERS

EXERCISE – I

1. 6

2. 2

4. (i) True, (ii) true (iii) true (iv) false

5. $P(n)$ is multiple of x^3