LESSON 3

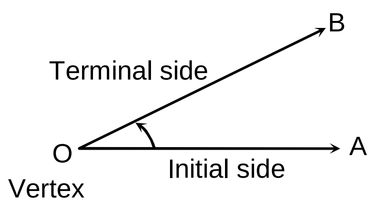
TRIGONOMETRIC FUNCTIONS & TRIGONOMETRIC EQUATIONS

Trigonometry {derived from Greek words 'trigon (triangle)' and 'metron (measurement)'} is that branch of Mathematics, which relates to the study of angles, measurement of angles, units of measurement. It also concerns itself with the six ratios, for a given angle and the relations satisfied by these ratios.

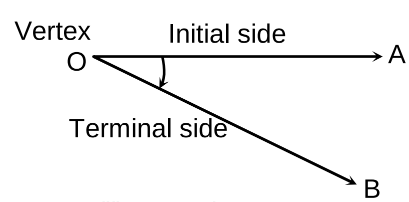
1. MEASUREMENT OF ANGLES

1.1 ANGLE

An angle is considered as figure obtained by rotating a given ray about its initial point. The original ray (before rotation) is known as **initial side** and the final position of the ray (after rotation) is known as **terminal side** of the angle, the point of rotation is known as **vertex**. The angle is said to be positive or negative as the rotation is anti-clockwise or clockwise.



(i) Positive angle



(ii) Negative angle

There are two methods used in measuring angles.

- **Sexagesimal System (Degree Measure)**

  The measure of an angle will be one degree, if the rotation from initial side to terminal side is $\frac{1}{360}^{th}$ of a revolution.

  1 right angle = 90 degrees (= 90°)

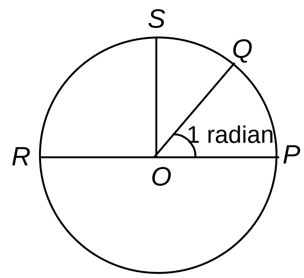
1 degree = 60 minutes (= 60')

1 minute = 60 seconds (= 60")

- **Circular System (Radian Measure)**

  One radian is the angle subtended at the centre of any circle by an arc of the circle equal in length to the radius. In fact it is a constant angle for otherwise it cannot be chosen as a unit of measurement.

1.2 RELATION BETWEEN DEGREES AND RADIANS



The angle subtended by a circle at its centre = $2\pi^c = 360°$

π radians = 180 degrees or $\pi^c = 180°$

$$1^c = \left(\frac{180}{\pi}\right)^0 = 57°16' \text{ (Approx)}$$

conversely from degrees to radians

$1° = (\pi/180)$ of a radian.

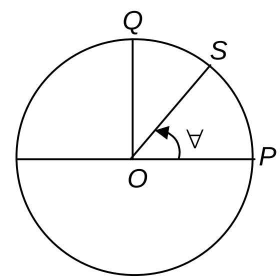
The unit radian is denoted by *c* (circular measure) and it is customary to omit this symbol *c.* Thus, when an angle is denoted as $\frac{\pi}{2}$, it means that the angle is $\frac{\pi}{2}$ radians where π is the number with approximate value 3.14159.

1.3 LENGTH OF AN ARC OF A CIRCLE

Consider an arc *PS* of a circle which subtends an angle θ (θ radians). Let $\angle POQ$ = 1 radian, then from definition of radian;

arc $PQ = r$

If the length of the arc $PS = l$, then



$$\frac{\angle POS}{\angle POQ} = \frac{\text{arc } PS}{\text{arc } PQ} \quad i.e., \quad \frac{\theta}{1} = \frac{l}{r} \quad \text{or} \quad l = r\theta;$$

Length of arc = (radius of the circle) ×
(angle subtended by that arc at the centre of circle)

Illustration 1

Question: **Find the radian measure corresponding to $37° 30'$.**

Solution: $\because$ $60' = 1°$ $\quad 30' = \left(\frac{1}{2}\right)^{\circ}$ $\quad 37° 30' = \left(37\frac{1}{2}\right)^{\circ} = \left(\frac{75}{2}\right)^{\circ}$

$\therefore$ 360° = 2π radians $\quad \left(\frac{75}{2}\right)^{\circ} = \left(\frac{2\pi}{360} \times \frac{75}{2}\right)$ radians $= \frac{5\pi}{24}$ radians

Illustration 2

***Question*:** **The minute hand of a clock is 10 cm long. How far does the tip of the hand move in 20 minutes ?**

Solution : The minute hand moves through 120° in 20 minutes or moves through $\frac{2⇔}{3}$ radians.

Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula $l = r∀ = 10 \mid \frac{2⇔}{3} X \frac{20⇔}{3}$ cm.

2. TRIGONOMETRIC FUNCTIONS OF AN ANGLE

The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle ∀, 0° < ∀ < 90° are defined as the ratios of two sides of a right-angled triangle with ∀ as one of the angles. However these can be defined through a unit circle more elegantly.

Draw a unit circle and take any two diameters at right angle as *X* and *Y* axes. Taking *OX* as the initial line, let $\overrightarrow{OP}$ be the radius vector corresponding to an angle ∀, where *P* lies on the unit circle. Let (*x, y*) be the coordinates of *P.*

Then by definition :

cos∀ = *x,* the *x* coordinate of *P*

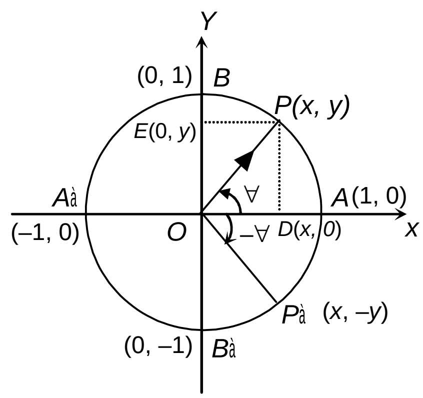
sin∀ = *y,* the *y* coordinate of *P*

$\tan∀ = \frac{y}{x}, x \mid 0$

$\cot∀ = \frac{x}{y}, y \mid 0$

$\sec∀ = \frac{1}{x},\ x \mid 0$

$\text{cosec}∀ = \frac{1}{y},\ y \mid 0$



Angles measured anticlockwise from the initial line *OX* are deemed to be positive and angles measured clockwise are considered to be negative.

Since we can associate a unique radius vector $\overline{OP}$ and a unique point *P* with each angle ∀, we say *x* and *y* and their ratios are functions of ∀. This justifies the term trigonometric 'function'. This definition holds good for all angles positive, negative, acute or not acute (irrespective of the magnitude of the angle).

This definition also helps us to write the sine and cosine of four important angles 0°, 90°, 180° and 270° easily.

∀ = 0°	*A*(1, 0)	cos 0 ⌈ X 1 and sin 0 ⌈ X 0
∀ = 90°	*B*(0, 1)	cos 90 ⌈ X 0 and sin 90 ⌈ X 1
∀ = 180°	*A*®(Z1, 0)	cos 180 ⌈ X Z1 and sin 180 ⌈ X 0
∀ = 270°	*B*®(0, Z1)	cos 270 ⌈ X 0 and sin 270 ⌈ X Z1

- Since, $\tan\theta = \frac{y}{x}, x \neq 0$, $\tan 90° = \frac{1}{0}$ and hence undefined. Similarly, sec 90°, cot 0°, cosec 0° are also undefined.
- 360° and 0° correspond to one and the same point $A(1, 0)$. Therefore, the trigonometric functions of 360° are the same as trigonometric functions of 0°. sin 360° = 0 and cos 360° = 1 and tan 360° = 0 Since θ, $2\pi + \theta$, $4\pi + \theta$, $6\pi + \theta$, ..., $2n\pi + \theta$ and $\theta - 2\pi$, $\theta - 4\pi$, $\theta - 6\pi$,....., $\theta - 2n\pi$ all correspond to the same radius vector, the trigonometric functions of all these angles are the same as those of θ.

$\therefore$ $\sin(2n\pi + \theta) = \sin\theta$ and $\sin(\theta - 2n\pi) = \sin\theta$

$\cos(2n\pi + \theta) = \cos\theta$ and $\cos(\theta - 2n\pi) = \cos\theta$

$\tan(2n\pi + \theta) = \tan\theta$ and $\tan(\theta - 2n\pi) = \tan\theta$

- **The range and sign of the trigonometric ratios in the four quadrants are depicted in the following table.**

Y

In the second quadrant (Only sine and cosecant are positive)			**In the first quadrant (All trigonometric ratios are positive)**		
sine	decreases from	1 to 0	sine	Increases from	0 to 1
cosine	decreases from	0 to -1	cosine	decreases from	1 to 0
tangent	increases from	$-\infty$ to 0	tangent	increases from	0 to ∞
contangent	decreases from	0 to $-\infty$	cotangent	decreases from	∞ to 0
secant	increases from	$-\infty$ to -1	secant	increases from	1 to ∞
cosecant	increases from	1 to ∞	cosecant	decreases from	∞ to 1

X′ O X

In the third quadrant (Only tangent and cotangent are positive)			**In the fourth quadrant (Only cosine and secant are positive)**		
sine	decreases from	0 to -1	sine	Increases from	-1 to 0
cosine	increases from	-1 to 0	cosine	increases from	0 to 1
tangent	increases from	0 to ∞	tangent	increases from	$-\infty$ to 0
contangent	decreases from	∞ to 0	cotangent	decreases from	0 to $-\infty$
secant	decreases from	-1 to $-\infty$	secant	decreases from	∞ to 1
cosecant	increases from	$-\infty$ to -1	cosecant	decreases from	-1 to $-\infty$

Y′

2.1 TRIGONOMETRIC FUNCTIONS OF $(-\theta)$

Let OP and OP' be the radii vectors, on the unit circle corresponding to θ and $-\theta$. If (x, y) are the coordinates of P, then $(x, -y)$ would be the coordinates of P'

Now $\sin\theta = y$ and $\sin(-\theta) = -y$. Hence, $\sin(-\theta) = -\sin\theta$

Similarly, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$

2.2 CIRCULAR FUNCTIONS OF ALLIED ANGLES

When θ is an acute angle then $90° - \theta$ is called the **complementary** angle of θ. When θ is acute, θ and $180° - \theta$ are called **supplementary** angles. Trigonometric functions of $(90° - \theta)$, $(180° - \theta)$ and other allied angles are related to trigonometric functions of θ as follows:

TABLE OF FORMULAE FOR ALLIED ANGLES

	$180° - \theta$	**$180° + \theta$**	**$360° - \theta$**	**$-\theta$**	**$90° - \theta$**	**$90° + \theta$**	**$270° - \theta$**	**$270° + \theta$**
Sin	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
Cos	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
Tan	$-\tan\theta$	$+\tan\theta$	$-\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

2.3 SOME IMPORTANT FACTS

The following may be noted

(i) For any power n, $(\sin A)^n$ is written as $\sin^n A$. Similarly for all trigonometric ratios.

(ii) cosec$A$, sec$A$ and cot$A$ are respectively the reciprocals of sin$A$, cos$A$ and tan$A$.

(iii) $\sin^2 A + \cos^2 A = 1$; $1 + \tan^2 A = \sec^2 A$ and $1 + \cot^2 A = \mathrm{cosec}^2 A$.

(iv) $\sec A - \tan A$ and $\sec A + \tan A$ are reciprocals. So also are $\mathrm{cosec} A - \cot A$ and $\mathrm{cosec} A + \cot A$.

(v) Whenever sec$A$ or tan$A$ is thought of for angle A, it is necessary to stress that, $A \neq \pi/2$ particularly, and generally $A \neq n\pi + \frac{\pi}{2}$ also in case of cot A and cosec A, $A \neq n\pi$.
($n \in$ ***N***, where ***N*** is the set of natural numbers).

(vi) $|\sin A| \leq 1 \Rightarrow -1 \leq \sin A \leq 1$
$|\cos A| \leq 1 \Rightarrow -1 \leq \cos A \leq 1$
$|\mathrm{cosec} A| \geq 1 \Rightarrow \mathrm{cosec} A \geq 1$ or $\mathrm{cosec} A \leq -1$
$|\sec A| \geq 1 \Rightarrow \sec A \geq 1$ or $\sec A \leq -1$

(vii) The trigonometric ratios are also called as trigonometric functions. They are also sometimes called circular functions.

(viii) The trigonometric functions, apart from possessing many other properties exhibit a property of the values being repeated when the angle is changed (increased or decreased) by a constant value. Such a property is referred to as periodicity.

Thus $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi)$
$= \sin(x - 2\pi) = \sin(x + 2k\pi)$, k an integer.
$\cos x = \cos(x + 2\pi) = \cos(x + 4\pi)$
$= \cos(x - 2\pi) = \cos(x + 2k\pi)$, k an integer.

Hence both sin$x$ and cos$x$ are periodic functions of period 2π radians. From (vi), it is clear that they are also bounded functions.

cosec x and sec x, whenever they exist, are also periodic of period 2π radians. tan x and cot x, when they exist, are periodic of period π radians. tan x, sec x, cosec x, cot x are unbounded functions.

(xi) Trigonometric ratios of 30°, 45° and 60° are of great importance in solving problems on heights and distances. These along with 0° and 90° are written in tabular form and remembered.

ANGLE / RATIO	0°	30°	45°	60°	90°
sine	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$2/\sqrt{3}$	1

Illustration 3

***Question*:** **Evaluate: cos(J3030°).**

Solution : cos(Z3030°) = cos(3030°) (using cos(Z∀) = cos∀) = cos(8 × 360° + 150°)

$$= \cos150° = \cos(180° \; Z \; 30°) = Z\cos30° = Z\frac{\sqrt{3}}{2}.$$

Illustration 4

***Question*:** **If cos ⁻ Z J $\frac{1}{2}$ and ® Y ⁻ Y $\frac{3®}{2}$, find the value of 4 tan² ⁻ J 3 cosec² ⁻.**

Solution : Since ∀ lies in the third quadrant, therefore sin ∀ is negative and tan ∀ is positive.

Now, $\sin∀ X \{ \sqrt{1 Z \cos^2 ∀} = Z\frac{\sqrt{3}}{2}$ $\quad$ $\text{cosec}\,∀ X \frac{Z2}{\sqrt{3}}$

And, $\tan∀ X \frac{\sin∀}{\cos∀} X \sqrt{3}$

Hence, $4\tan^2 ∀ Z 3\,\text{cosec}^2 ∀ X 4 \,|\, 3 \, Z \, 3 \,|\, \frac{4}{3} X 8$

3. CIRCULAR FUNCTIONS OF COMPOUND ANGLES

3.1 ADDITION AND SUBTRACTION FORMULAE

To prove:

) **cos(¯ + ¤) = cos¯ cos¤ J sin¯ sin¤** ...(i)

Consider a unit circle with origin as its centre.

Let P_4OP_1 = ∀ and P_1OP_2 = ←

… P_4OP_2 = ∀ + ←

P_4OP_3 = Z←

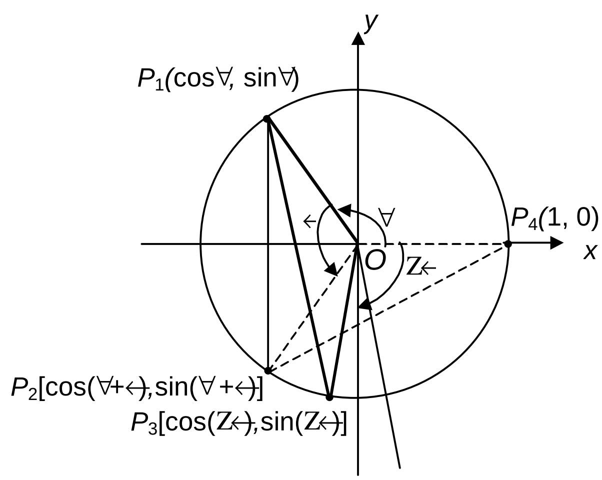
Coordinate of P_1, P_2, P_3, P_4 are

P_1(cos∀, sin∀)

P_2[cos(∀ + ←), sin(∀ + ←)]

P_3[cos(Z←), sin(Z←)]

P_4(1, 0)



ζ P_1O P_3 is congruent to ζ P_2O P_4

∵ $OP_1 = OP_4 = OP_3 = OP_2$ = Radius of the circle

P_1OP_3 = P_2OP_4 = 360 Z (∀ + ←)

… By Side Angle Side, the triangles are congruent.

… $P_1P_3 = P_2P_4$

Applying the distance formula,

$P_1P_3^2$ = [cos∀ Z cos(Z←)]^2 + [sin∀ Z sin(Z←)]^2

= (cos∀ Z cos←)^2 + (sin∀ + sin ←)^2

[using cos(Z←) = cos← and sin(Z←) = Zsin←]

= cos^2∀ + cos^2← Z 2cos∀ cos← + sin^2∀ + sin^2← + 2sin∀ sin←

= 2 Z 2 (cos∀ cos← Z sin∀ sin←)

$P_2P_4^2$ = [1 Z cos(∀ + ←)]^2 + [0 Z sin(∀ + ←)]^2

= 1 Z 2cos(∀ + ←) + cos^2(∀ + ←) + sin^2(∀ + ←) = 2 Z 2cos(∀ + ←).

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2$ X $P_2P_4^2$.

… 2 Z 2 (cos∀ cos← Z sin∀ sin←) = 2 Z 2cos(∀ + ←)

Hence cos(∀ + ←) = cos∀ cos← Z sin∀ sin←

) **cos(¯ J ¤) = cos¯ cos¤ + sin¯ sin¤** ...(ii)

Replacing ← by Z← in identity (i), we get

cos(∀ Z ←) = cos∀ cos(Z←) Z sin∀ sin (Z←)

or $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\Rightarrow$ $\mathbf{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$ ...(iii)

$\therefore$ $\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) = \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$

$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\Rightarrow$ $\mathbf{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$ ...(iv)

To get the result, replace β by $-\beta$ in (iii)

$\Rightarrow$ $\mathbf{\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$ ...(v)

$\because$ $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

Dividing numerator and denominator by $\cos\alpha\cos\beta$, we get

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\Rightarrow$ $\mathbf{\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$ ...(vi)

To get the result, replace β by $-\beta$ in (v)

In (v) and (vi) none of the angle α, β, $\alpha + \beta$ and $\alpha - \beta$ is an odd multiple of $\frac{\pi}{2}$

$\Rightarrow$ $\mathbf{\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}}$ ...(vii)

$\Rightarrow$ $\mathbf{\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}}$ ...(viii)

In (vii) and (viii) none of the angle α, β, $\alpha + \beta$ and $\alpha - \beta$ is a multiple of π

3.2 FUNCTION OF (*A* + *B* + *C*)

1. $\sin(A + B + C) = \Sigma \sin A\cos B\cos C - \sin A\sin B\sin C$

Proof :

$\sin\{A + (B + C)\} = \sin A \,.\, \cos(B + C) + \cos A.\sin(B + C)$

$= \sin A\{\cos B\cos C - \sin B\sin C\} + \cos A\{\sin B\cos C + \cos B\sin C\}$

$= \Sigma \sin A\cos B\cos C - \sin A\sin B\sin C$ ...(iii)

2. $\cos(A + B + C) = \cos A\cos B\cos C - \Sigma \cos A\sin B\sin C$

Proof is similar to that of the previous formula.

3. $\tan(A + B + C) = \dfrac{\sum \tan A - \tan A \tan B \tan C}{1 - \sum \tan B \tan C}$.

Proof :

$$\tan(A + B + C) = \tan\{A + (B + C)\} = \frac{\tan A + \tan(B + C)}{1 - \tan A.\tan(B + C)} = \frac{\tan A + \dfrac{\tan B + \tan C}{1 - \tan B \tan C}}{1 - \tan A.\dfrac{\tan B + \tan C}{1 - \tan B \tan C}}$$

$$= \frac{(\tan A + \tan B + \tan C) - \tan A.\tan B.\tan C}{1 - (\tan A \tan B + \tan B \tan C + \tan C \tan A)} = \frac{S_1 - S_3}{1 - S_2}$$

where S_r = sum of the products (of tangents) taken r at a time.

Illustration 5

***Question*:** **If $\sin\alpha = \dfrac{8}{17}$ and $\cos\beta = \dfrac{9}{41}$, find $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, where α is an obtuse angle and β is an acute angle.**

Solution : Since $\sin\alpha = \dfrac{8}{17}$, $\cos^2\alpha = 1 - \dfrac{64}{289} = \dfrac{225}{289}$

$\therefore$ $\cos\alpha = \pm\dfrac{15}{17}$. As α is obtuse, $\cos\alpha$ is negative.

$\therefore$ $\cos\alpha = -\dfrac{15}{17}$

$\sin^2\beta = 1 - \dfrac{81}{1681} = \dfrac{1600}{1681}$

$\therefore$ $\sin\beta = \pm\dfrac{40}{41}$

As β is acute, $\sin\beta$ is positive

$\therefore \sin\beta = +\dfrac{40}{41}$

Now $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \dfrac{8}{17}.\dfrac{9}{41} + \left(-\dfrac{15}{17}\right)\left(\dfrac{40}{41}\right) = \dfrac{72 - 600}{697} = -\dfrac{528}{697}$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\dfrac{15}{17}\right)\left(\dfrac{9}{41}\right) - \left(\dfrac{8}{17}\right)\left(\dfrac{40}{41}\right) = -\dfrac{455}{697}$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(\dfrac{8}{17}\right)\left(\dfrac{9}{41}\right) - \left(-\dfrac{15}{17}\right)\left(\dfrac{40}{41}\right) = \dfrac{672}{697}$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \left(-\dfrac{15}{17}\right)\left(\dfrac{9}{41}\right) + \left(\dfrac{8}{17}\right)\left(\dfrac{40}{41}\right) = \dfrac{185}{697}$.

Illustration 6

***Question*:** **Prove that tan 112A tan 99A tan 13A = tan112A − tan 99A − tan13A.**

Solution : In this problem, note that one of the three angles (112A) is the sum of the other two (99A + 13A)

$112A = 99A + 13A$

$\tan 112A = \tan(99A + 13A) = \frac{\tan 99A + \tan 13A}{1 - \tan 99A \tan 13A}$

$\tan 112A\,(1 - \tan 99A \tan 13A) = \tan 99A + \tan 13A$

$\therefore \quad \tan 112A - \tan 99A - \tan 13A = \tan 112A \tan 99A \tan 13A$

3.3 MULTIPLE ANGLE FORMULAE

3.3.1 Functions of 2A

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(iii) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

These are special cases of the addition formulae by taking $B = A$. The formulae for $\cos 2A$ leads to two results whose application occurs often in problems.

$1 + \cos 2A = 2\cos^2 A$ and $1 - \cos 2A = 2\sin^2 A$

Besides these, $\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$ and $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$; $A \neq (2n + 1)\frac{\pi}{2}$

If A be replaced by $\frac{A}{2}$, these formulae reduce to

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(iii) $\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

3.3.2 Functions of 3A

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

Proof:

$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos^2 A + \sin A\,(1 - 2\sin^2 A) = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

Proof :

$\cos 3A = \cos(2A + A) = \cos 2A \,.\, \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A$

$= (2\cos^2 A - 1)\cos A - 2\cos A\,(1 - \cos^2 A)$

$= 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$; $A \neq (2n+1)\dfrac{\pi}{2}$, $A \neq n\pi \pm \dfrac{\pi}{6}$

Proof:

$$\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan^2 A}{1-\tan^2 A}} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Note: This formula can be derived from the expansion of $\tan(A + B + C) = \dfrac{S_1 - S_3}{1 - S_2}$ by taking $B = C = A$.

Illustration 7

Question: **Prove that** $\dfrac{\tan(\pi/4 + A)}{\tan(\pi/4 - A)} = \dfrac{2\cos A + \sin A + \sin 3A}{2\cos A - \sin A - \sin 3A}$.

Solution :

$$\text{L.H.S.} = \frac{\dfrac{1+\tan A}{1-\tan A}}{\dfrac{1-\tan A}{1+\tan A}} = \frac{(1+\tan A)^2}{(1-\tan A)^2} = \frac{1+\tan^2 A + 2\tan A}{1+\tan^2 A - 2\tan A}$$

$$= \frac{1 + \dfrac{2\tan A}{1+\tan^2 A}}{1 - \dfrac{2\tan A}{1+\tan^2 A}} = \frac{1+\sin 2A}{1-\sin 2A}$$

$$\text{R.H.S.} = \frac{2\cos A + 2\sin 2A.\cos A}{2\cos A - 2\sin 2A\cos A} = \frac{2\cos A(1+\sin 2A)}{2\cos A(1-\sin 2A)} = \frac{1+\sin 2A}{1-\sin 2A}$$

Both sides reduce to the same result.

3.4 EXPRESSING PRODUCTS OF TRIGONOMETRIC FUNCTIONS AS SUM OR DIFFERENCE

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$ ...(i)

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

The above four formulae can be obtained by expanding the right hand side and simplifying.

Note : In the fourth formula, there is a change in the pattern. Angle $(A - B)$ comes first and $(A + B)$ later. In the first quadrant, the greater the angle, the less the cosine. Hence cosine of the smaller angle is written first [to get a positive result].

Illustration 8

Question: **Show that 8 sin 10° sin 50° sin 70° = 1.**

Solution : L.H.S. = 4 (2sin 50° sin 10°) sin70°

$= 4\{\cos(50° - 10°) - \cos(50° + 10°)\}\sin 70°$, using 2 sin$A$ sin$B$

$= 2\{\sin 110° + \sin 30°\} - 2\sin 70°$ since cos60° = 1/2.

$= \cos(A - B) - \cos(A + B)$

$= 4(\cos 40° - \cos 60°)\sin 70°$

$= 2.(2\sin 70°.\cos 40°) - 4\cos 60°\sin 70°$

$= 2\sin 70° + 2\sin 30° - 2\sin 70°$

$= 2\sin 30° = 1.$

3.5 EXPRESSING SUM OR DIFFERENCE OF TWO SINES OR TWO COSINES AS A PRODUCT

In the formulae derived in the earlier section if we put $A + B = C$ and $A - B = D$, then $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$, these formulae can be rewritten as

$$\sin C + \sin D = 2\sin\frac{C+D}{2}.\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}.\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}.\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}$$

Note carefully the order of the cosines in the last formula. It is presumed that $C > D$.

Illustration 9

Question: **Show that** $\frac{\sin 7x - \sin 3x - \sin 5x + \sin x}{\cos 7x + \cos 3x - \cos 5x - \cos x} = \tan 2x.$

Solution : Numerator = $(\sin 7x + \sin x) - (\sin 5x + \sin 3x)$

$= 2\sin 4x\ .\cos 3x - 2\sin 4x\ .\cos x$ {using C.D. formula}

$= 2\sin 4x(\cos 3x - \cos x)$

Denominator $= (\cos 3x - \cos 5x) - (\cos x - \cos 7x)$

$= 2\sin 4x \sin x - 2\sin 4x\sin 3x = 2\sin 4x(\sin x - \sin 3x)$

$\therefore$ the given expression $= \frac{\cos 3x - \cos x}{\sin x - \sin 3x} = \frac{\cos x - \cos 3x}{\sin 3x - \sin x} = \frac{2\sin 2x \sin x}{2\cos 2x \sin x} = \tan 2x.$

3.6 SOME MORE RESULTS

- $\sin(A + B) \times \sin(A - B) = \sin^2 A - \sin^2 B$
- $\cos(A + B) \times \cos(A - B) = \cos^2 A - \sin^2 B$
- $\sin 18^\circ = \dfrac{(\sqrt{5} - 1)}{4} = \cos 72^\circ$
- $\cos 18^\circ = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72^\circ$
- $\cos 36^\circ = \dfrac{(\sqrt{5} + 1)}{4} = \sin 54^\circ$
- $\sin 36^\circ = \dfrac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54^\circ$
- $\tan 22\dfrac{1}{2}^\circ = (\sqrt{2} - 1)$

3.6.1 Triple Angle Formulae

- $\sin\theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \dfrac{1}{4}\sin 3\theta$
- $\cos\theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \dfrac{1}{4}\cos 3\theta$
- $\tan\theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$

Illustration 10

Question: **Using the values of trigonometric functions for 18° and 36°, prove the following:**

(i) $\sin^2 72^\circ - \sin^2 60^\circ = \dfrac{\sqrt{5} - 1}{8}$ **(ii)** $\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$

Solution: (i) L.H.S. $= \sin^2 72^\circ - \sin^2 60^\circ$

$$= [\sin(90^\circ - 18^\circ)]^2 - \frac{3}{4} = \cos^2 18^\circ - \frac{3}{4} = \frac{10 + 2\sqrt{5}}{16} - \frac{3}{4} = \frac{2\sqrt{5} - 2}{16}$$

$$= \frac{\sqrt{5} - 1}{8} = \text{R.H.S}$$

(ii) L.H.S. $= \sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}.\sin\dfrac{3\pi}{5}.\sin\dfrac{4\pi}{5} = \sin\dfrac{\pi}{5}.\sin\dfrac{2\pi}{5}.\sin\left(\pi - \dfrac{2\pi}{5}\right)\sin\left(\pi - \dfrac{\pi}{5}\right)$

$$= \sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \sin^2\frac{\pi}{5}\left[\sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)\right]^2 = \sin^2\frac{\pi}{5}.\cos^2\frac{\pi}{10}$$

$$= \frac{10 - 2\sqrt{5}}{16}.\frac{10 + 2\sqrt{5}}{16} \quad \text{as } \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \text{ and } \sin 36^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$= \frac{100 - 20}{16 \times 16} = \frac{80}{16 \times 16} = \frac{5}{16}$$

Illustration 11

Question: **Prove that $(\cos A + \cos B)^2 + (\sin A - \sin B)^2 = 4\cos^2\left(\frac{A+B}{2}\right)$.**

Solution : L.H.S. $= (\cos A + \cos B)^2 + (\sin A - \sin B)^2$

$$= (\cos^2 A + \cos^2 B + 2\cos A\cos B) + (\sin^2 A + \sin^2 B - 2\sin A\sin B)$$

$$= (\cos^2 A + \sin^2 B) + (\cos^2 B + \sin^2 B) + 2(\cos A\cos B - \sin A\sin B)$$

$$= 2 + 2\cos(A + B) = 2.2\cos^2\frac{A+B}{2} = \text{R.H.S.}$$

4. CONDITONAL TRIGONOMETRICAL IDENTITIES

4.1 IDENTITIES

A trigonometric equation is an identity if it is true for all values of the angle or angles involved.

4.2 CONDITIONAL IDENTITIES

When the angles involved satisfy a given relation, the identity is called conditional identity. In proving these identities we require properties of complementary and supplementary angles.

4.3 SOME IMPORTANT CONDITIONAL IDENTITIES

If $A + B + C = \pi$ then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, where $A, B, C \neq (2n+1)\frac{\pi}{2}, n \in I$
- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$, where $A, B, C \neq n\pi, n \in I$
- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
- $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$, where $A, B, C \neq (2n+1)\pi, n \in I$
- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}\cot\frac{C}{2}$, where $A, B, C \neq 2n\pi, n \in I$

Illustration 12

Question: **If $A + B + C = 2\pi$, prove that $\cos^2 B + \cos^2 C - \sin^2 A = 2\cos A\cos B\cos C$.**

Solution : L.H.S. = $\cos^2 B + \cos^2 C - \sin^2 A$

$$= \frac{1}{2}[\cos 2B + 1 + \cos 2C + 1 + \cos 2A - 1]$$

$$= \frac{1}{2}[\cos 2A + \cos 2B + \cos 2C + 1] = \frac{1}{2}[2\cos(A+B)\cos(A-B) + 2\cos^2 C]$$

$= \cos(\pi - C)\cos(A - B) + \cos^2 C$

$= \cos C[\cos(A - B) + \cos(\pi - \overline{A+B})]$

$= \cos C[\cos(A - B) + \cos(A + B)]$

$= 2\cos A \cos B \cos C$

= R.H.S.

Illustration 13

Question: **If $A + B + C = 180°$, prove that**

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

(ii) $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$.

Solution : $B + C = 180° - A$

$\therefore$ $\sin(B + C) = \sin(180° - A) = \sin A$ and $\cos(B + C) = \cos(180° - A) = -\cos A$ ...(i)

(i) $\sin 2A + \sin 2B + \sin 2C = 2\sin A \cos A + 2\sin(B + C) . \cos(B - C)$ {using C-D formula}

$= 2\sin A \cos A + 2\sin A \cos(B - C)$ {using equation (i)}

$= 2\sin A \{\cos A + \cos(B - C)\}$

$= 2\sin A \{-\cos(B + C) + \cos(B - C)\}$ (using (i) to change $\cos A$)

$= 2\sin A . 2\sin B \sin C$ (using C − D formula)

$= 4\sin A \sin B \sin C$

(ii) $\sin 2A + \sin 2B - \sin 2C = 2\sin A \cos A + 2\cos(B + C)\sin(B - C)$

$= 2\sin A \cos A - 2\cos A \sin(B - C) = 2\cos A \{\sin A - \sin(B - C)\}$

$= 2\cos A \{\sin(B + C) - \sin(B - C)\}$ {using equation (i)}

$= 2\cos A . (2\cos B \sin C) = 4\cos A \cos B \sin C$

5. BOUNDS OF THE EXPRESSION $a\cos\theta + b\sin\theta$

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$$

$$= \sqrt{a^2+b^2}\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= \sqrt{a^2+b^2}\,\sin(\theta + \alpha), \text{ where } \tan\alpha = \frac{a}{b}$$

Also, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\,\cos(\theta-\varphi)$, where $\tan\varphi = \dfrac{b}{a}$

$$-1 \le \sin(\theta+\alpha) \le 1$$

and hence, $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Thus the expression $a\cos\theta + b\sin\theta$ is bounded above by $\sqrt{a^2+b^2}$ and bounded below by $-\sqrt{a^2+b^2}$.

Illustration 14

Question: **Prove that 5 cos*x* + 3cos(*x* + π/3) + 3 lies between − 4 and 10.**

Solution : $5\cos x + 3\left[\cos x\cos\dfrac{\pi}{3} - \sin x\sin\dfrac{\pi}{3}\right] + 3$

$$= \cos x\left(5+\frac{3}{2}\right) - \sin x.\frac{3\sqrt{3}}{2} + 3 = \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \sqrt{\frac{169}{4}+\frac{27}{4}}\left[\frac{13/2}{\sqrt{\frac{169}{4}+\frac{27}{4}}}\cos x - \frac{3\sqrt{3}/2}{\sqrt{\frac{169}{4}+\frac{27}{4}}}\sin x\right] + 3$$

$= 7(\cos\alpha\cos x - \sin\alpha\sin x) + 3$ (where $\tan\alpha = \dfrac{3\sqrt{3}}{13}$)

$= 7\cos(\alpha + x) + 3$

$\therefore\ -1 \le \cos(\alpha + x) \le 1$

i.e., $-7 + 3 \le 7\cos(\alpha + x) + 3 \le 7 + 3$ *i.e.,* $-4 \le 7\cos(\alpha + x) + 3 \le 10$

6. GRAPH OF TRIGONOMETRIC FUNCTIONS

6.1 DOMAIN AND RANGE

Let $y = f(x)$ be a function, then the set of values of x satisfying it is known as domain set and the set of values of y so obtained is known as range of function.

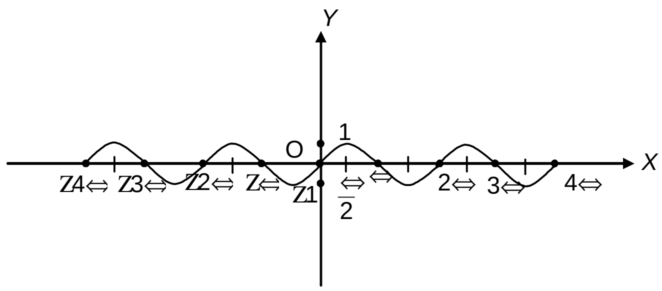
6.2 Graph of $y = \sin x$

From the knowledge of trigonometry we compute the following table :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Domain : *R*

Range : [–1, 1]

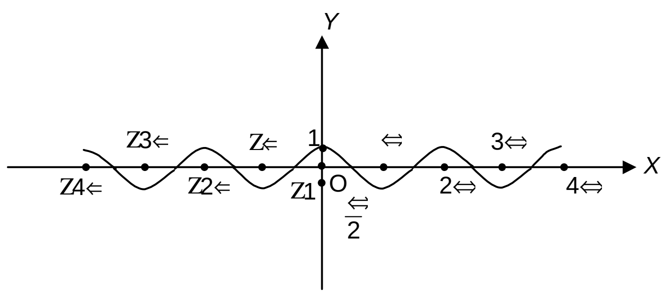


$y = \sin x$

6.3 Graph of $y = \cos x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

Domain : R
Range : [–1, 1]



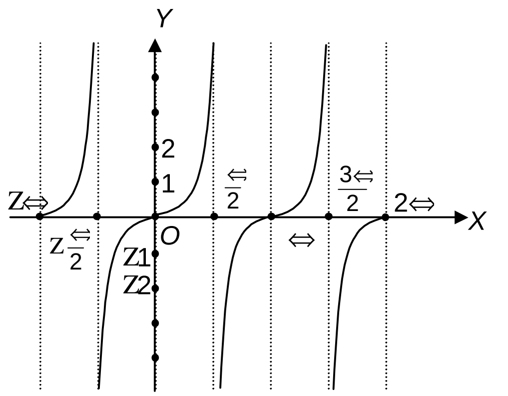
$y = \cos x$

6.4 Graph of $y = \tan x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Domain : $R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$

Range : R



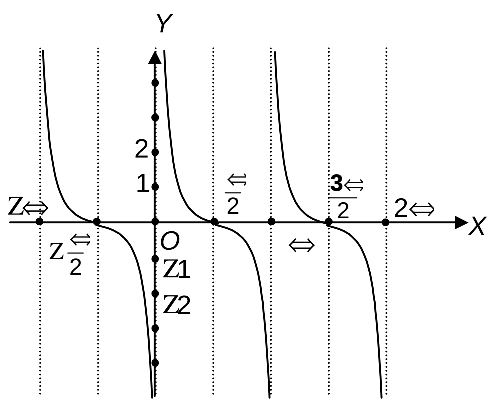
$y = \tan x$

6.5 Graph of $y = \cot x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cot$x$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	undefined

Domain : $R - \{n\pi : n \in I\}$

Range : R



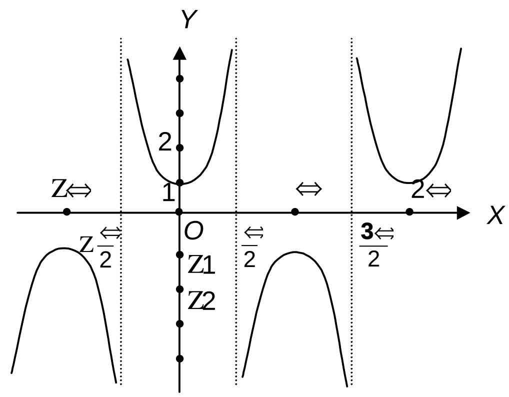
$y = \cot x$

6.6 Graph of $y = \sec x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sec$x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	-2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	-1

Domain : $R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$

Range : $(-\infty, -1] \cup [1, \infty)$



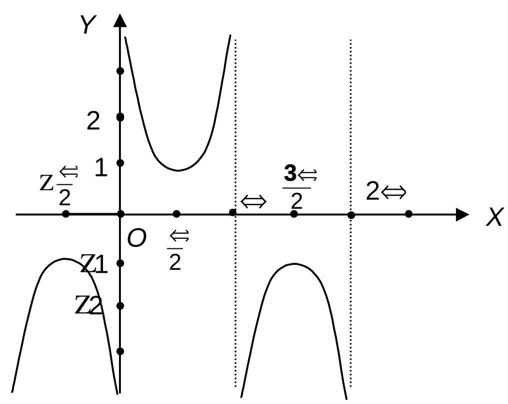
$y = \sec x$

6.7 Graph of $y = \operatorname{cosec} x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cosec$x$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined

Domain : $R - \{n\pi, n \in I\}$

Range : $(-\infty, -1] \cup [1, \infty)$

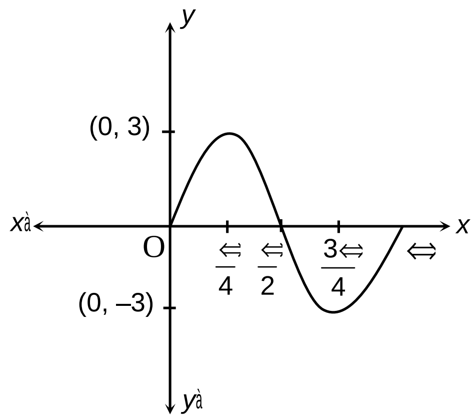


$y = \operatorname{cosec} x$

Illustration 15

Question: **Sketch the graph of $y = 3\sin 2x$.**

Solution : To obtain the graph of $y = 3\sin 2x$ we first draw the graph of $y = \sin x$ in the interval $[0, 2\pi]$ and then divide the x-coordinates of the points where it crosses x-axis by 2. The maximum and minimum values are 3 and –3 respectively as shown in figure.



Important formulae/points

- $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$ for all values of θ.

Function	Domain	Range
• $y = \sin x$	***R***	$[-1, 1]$
• $y = \cos x$	***R***	$[-1, 1]$
• $y = \tan x$	***R*** $-(2n+1)\frac{\pi}{2}$	***R***
• $y = \operatorname{cosec} x$	***R*** $- n\pi$	$(-\infty, -1] \cup [1, \infty)$
• $y = \sec x$	***R*** $-(2n+1)\frac{\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$
• $y = \cot x$	***R*** $- n\pi$	***R***

7. BASIC TRIGONOMETRIC EQUATIONS

Equations in which one or more than one trigonometric functions are involved are known as trigonometric equations.

Consider the following :

$\cos\theta - \sin\theta = 1$ ...(i)

$|\sec(\theta - \pi/4)| = 2$ ...(ii)

$\sin^2\theta + \cos^2\theta = 1$ ...(iii)

Equation (i) is satisfied if we put $\theta = 0, 2\pi, 4\pi, \ldots\ldots$ *etc.* in it. Equation (ii) is satisfied if we put $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, -\frac{\pi}{12}$, *etc.* in it. But equation (iii) is satisfied for any value of θ. Equations (i) & (ii) are called **trigonometric equations** while (iii) is a **trigonometric identity**.

A trigonometric equation has three **kinds of solutions:**

1. Principal solution : Numerically smallest value of the unknown angle satisfying the given equation.
2. Particular solution : Any value of angle satisfying the given equation.
3. General solution : Collection of all particular solutions.

7.1 METHOD TO FIND PRINCIPAL VALUE (NUMERICALLY LEAST ANGLE)

- First draw a trigonometric – circle and mark the quadrant in which the angle may lie.
- Select anticlockwise direction for 1st and 2nd quadrants and select clockwise for 3rd and 4th quadrants.
- Find the angle in the first rotation.
- Select the numerically least angle from these two values; the angle thus found will be the principal value.
- In case, two angles (one with positive sign and the other with negative sign) qualify for the numerically least angle then we select the angle with positive sign as the principal value. *e.g.*, Principal solution of $\sec\theta = 2$ is $\theta = \pi/3$, although $\theta = -\pi/3$ also satisfies it.

Note: In questions of **principal solutions** we have to find the values of unknown angle belongs to $[0, 2\pi)$, whereas in case of **principal solution** follow the above method.

Illustration 16

Question: **Find the principal solutions of following equations:**

(i) $\tan\theta = \frac{1}{\sqrt{3}}$ **(ii)** $\sec\theta = 2$ **(iii)** $\cot\theta = -\sqrt{3}$ **(iv)** $\text{cosec}\,\theta = -2$

Solution: **(i)** $\because$ $\tan\theta$ is positive in Ist and IIIrd quadrant.

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan\left(\pi + \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Also, $\frac{\pi}{6}, \pi + \frac{\pi}{6} \in [0, 2\pi)$

$\therefore$ Principal solutions are $\frac{\pi}{6}, \frac{7\pi}{6}$.

(ii) $\sec\theta$ is positive in Ist and IVth quadrant.

$$\sec\frac{\pi}{3} = 2 \text{ and } \sec\left(2\pi - \frac{\pi}{3}\right) = 2$$

Also, $\frac{\pi}{3}, 2\pi - \frac{\pi}{3} \in [0, 2\pi)$

$\therefore$ Principal solutions are $\frac{\pi}{3}, \frac{5\pi}{3}$.

(iii) $\cot\theta$ is negative in IInd and IVth quadrant.

Now, $\cot\frac{\pi}{6} = \sqrt{3}$ $\cot\left(\pi - \frac{\pi}{6}\right) = -\sqrt{3}$ and $\cot\left(2\pi - \frac{\pi}{6}\right) = -\sqrt{3}$

$\cot\frac{5\pi}{6} = -\sqrt{3}$ and $\cot\frac{11\pi}{6} = -\sqrt{3}$

Also, $\frac{5\pi}{6}, \frac{11\pi}{6} \in [0, 2\pi]$

$\therefore$ Principal solutions are $\frac{5\pi}{6}, \frac{11\pi}{6}$

(iv) $\therefore$ cosec θ is negative in IIIrd and IVth quadrant.

Now, $\operatorname{cosec}\frac{\pi}{6} = 2$ $\quad \operatorname{cosec}\left(\pi + \frac{\pi}{6}\right) = -2$ and $\operatorname{cosec}\left(2\pi - \frac{\pi}{6}\right) = -2$

$\operatorname{cosec}\frac{7\pi}{6} = -2$ and $\operatorname{cosec}\frac{11\pi}{6} = -2$

Also, $\frac{7\pi}{6}, \frac{11\pi}{6} \in [0, 2\pi]$

$\therefore$ Principal solutions are $\frac{7\pi}{6}, \frac{11\pi}{6}$.

7.2 GENERAL SOLUTION OF SOME SIMPLE TRIGONOMETRIC EQUATIONS

We list below the general solution of some simple trigonometric equations. ($n \in I$ in this lesson)

(i) $\sin\theta = 0$ $\qquad \theta = n\pi$

(ii) $\cos\theta = 0$ $\qquad \theta = (2n + 1)\frac{\pi}{2}$

(iii) $\tan\theta = 0$ $\qquad \theta = n\pi$

(iv) $\sin\theta = 1$ $\qquad \theta = (4n + 1)\frac{\pi}{2}$

(v) $\sin\theta = -1$ $\qquad \theta = (4n + 3)\frac{\pi}{2}$

(vi) $\cos\theta = 1$ $\qquad \theta = 2n\pi$

(vii) $\cos\theta = -1$ $\qquad \theta = (2n + 1)\pi$

Proof :

(i) **$\sin\theta = 0$**

from the adjoining figure,

$\sin\theta = \frac{y}{r}$ $\qquad \Rightarrow y = 0$

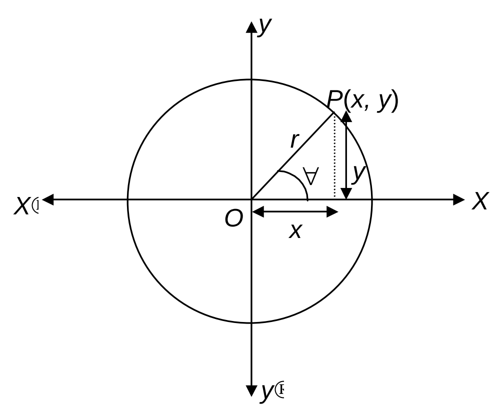
$\therefore$ *OP* coincide with the *x* axis.

If *OP* coincides with *OX*, then

$\theta = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots\ldots\ldots\ldots$

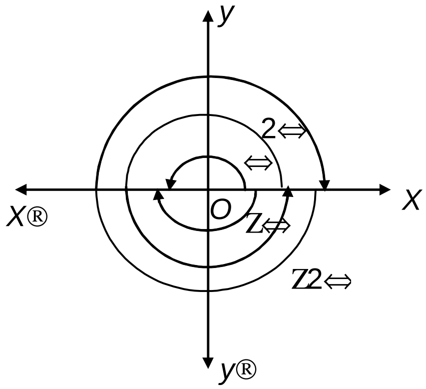
If *OP* coincides with *OX′*, then

$\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots\ldots\ldots\ldots$



Generalising,

$\boldsymbol{\theta = n\pi}$, where, $n = 0, \pm 1, \pm 2, ...$



(ii) $\cos\theta = 0$

$x/r = 0$

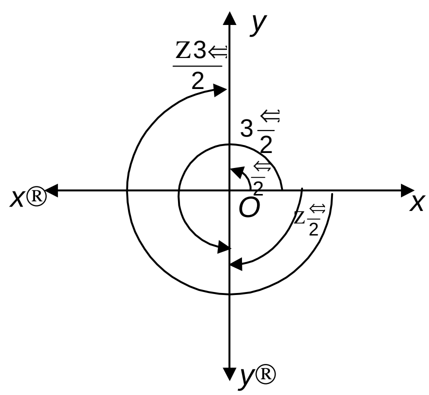
$x = 0$

OP coincides with the y-axis. If *OP* coincides with *OY*,

$\theta = \pm \pi/2, \pm 5\pi/2, ...$

If *OP* coincides with *OY'*,

$\theta = \pm 3\pi/2, \pm 7\pi/2, ...$



Generalising,

$\boldsymbol{\theta = (2n + 1)\pi/2}$, where $n = 0, \pm 1, \pm 2, ...$

(iii) $\tan\theta = 0$

$y/x = 0$ $\quad y = 0$

A case similar to $\sin\theta = 0$

$\therefore \quad \boldsymbol{\theta = n\pi}$, where $n = 0, \pm 1, \pm 2, ...$

Illustration 17

Question: **Solve $\sin x + \sin 3x + \sin 5x = 0$.**

Solution: $\sin x + \sin 3x + \sin 5x = 0$ or $2\sin 3x \cos 2x + \sin 3x = 0$ or $\sin 3x(2\cos 2x + 1) = 0$

$\Rightarrow \quad \sin 3x = 0$ or $2\cos 2x + 1 = 0$

Now, $\sin 3x = 0 \Rightarrow 3x = m\pi, m \in I \Rightarrow x = \dfrac{m\pi}{3}, m \in I$

Also, $2\cos 2x + 1 = 0$ implies $\cos 2x = -\dfrac{1}{2} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{2\pi}{3}$

$\therefore \quad 2x = 2n\pi \pm \dfrac{2\pi}{3}, n \in I$ or $x = n\pi \pm \dfrac{\pi}{3} = (3n \pm 1)\dfrac{\pi}{3}$

These solutions are included in the solution $x = \frac{m\pi}{3}, m \in I$ because $3n \pm 1$ is also an integer.

$\therefore$ The solution is $x = \frac{m\pi}{3}, m \in I$

7.3 GENERAL SOLUTION OF $\sin\theta = \sin\alpha$

$\theta = n\pi + (-1)^n\alpha,\ n \in I$

Proof :

$\sin\theta = \sin\alpha$

$\sin\theta - \sin\alpha = 0 \Rightarrow 2\cos\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0.$

Case I:

$\cos\frac{(\theta+\alpha)}{2} = 0 \Rightarrow \frac{(\theta+\alpha)}{2} = (2n+1)\pi/2$

where, $n = 0, \pm 1, \pm 2, ...$

$(\theta + \alpha) = (2n+1)\pi \Rightarrow \theta = (2n+1)\pi - \alpha \Rightarrow \theta = (2n+1)\pi + (-1)^{2n+1}\alpha$

Case II:

$\sin\frac{(\theta-\alpha)}{2} = 0 \Rightarrow \frac{(\theta-\alpha)}{2} = n\pi$, where $n \in I$.

$\theta = (2n\pi + \alpha) \Rightarrow \theta = 2n\pi + (-1)^{2n}\alpha$

Combining case I and case II,

$\theta = n\pi + (-1)^n\alpha$, where $n \in I$.

Illustration 18

Question: **Solve the equations:**

(i) cosec $m\theta = 2$ (ii) $\sin 2\theta = \frac{\sqrt{5}-1}{4}$

Solution: (i) $\text{cosec}\, m\theta = 2$

or $\sin m\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore m\theta = n\pi + (-1)^n\frac{\pi}{6}, n \in I \Rightarrow \theta = \frac{n\pi}{m} + (-1)^n\frac{\pi}{6m}, n \in I$

(ii) $\sin 2\theta = \frac{\sqrt{5}-1}{4}$

$\therefore \sin 2\theta = \sin 18^\circ = \sin\frac{\pi}{10}$

$\therefore 2\theta = n\pi + (-1)^n\frac{\pi}{10}, n \in I \Rightarrow \theta = \frac{n\pi}{2} + (-1)^n\frac{\pi}{20}, n \in I$

7.4 GENERAL SOLUTION OF $\cos\theta = \cos\alpha$

$\theta = 2n\pi \pm \alpha,\ n \in I$

Proof :

$\cos\theta = \cos\alpha$

$$\cos\theta - \cos\alpha = 0$$

$$-2\sin\frac{(\theta+\alpha)}{2}\sin\frac{(\theta-\alpha)}{2} = 0.$$

Case I:

$$\sin\frac{(\theta+\alpha)}{2} = 0$$

$$\frac{(\theta+\alpha)}{2} = n\pi \qquad \theta = (2n\pi - \alpha), \text{ where } n \in I.$$

Case II:

$$\sin\frac{(\theta-\alpha)}{2} = 0 \qquad \frac{(\theta-\alpha)}{2} = n\pi \qquad \theta = (2n\pi + \alpha) \text{ where } n \in I.$$

Combining case I and case II,

$$\theta = 2n\pi \pm \alpha, \quad \text{where } n \in I.$$

Illustration 19

Question: **Solve the equations:**

(i) $\cos 3\theta = \frac{\sqrt{3}}{2}$ **(ii)** $\sec n\theta = -\sqrt{2}$

Solution: (i) $\cos 3\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$$3\theta = 2n\pi \pm \frac{\pi}{6},\ n \in I.$$

(ii) $\sec n\theta = -\sqrt{2}$

$$\cos n\theta = -\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\therefore \quad \cos n\theta = \cos\frac{3\pi}{4} \qquad n\theta = 2k\pi \pm \frac{3\pi}{4},\ k \in I$$

$$\theta = \frac{2k\pi}{n} \pm \frac{3\pi}{4n},\ k \in I$$

7.5 GENERAL SOLUTION OF $\tan\theta = \tan\alpha$

$\theta = n\pi + \alpha,\ n \in I$, where $\alpha \neq (2m+1)\frac{\pi}{2},\ \forall m \in I$

Proof :

tan¯ = tan~

tan∀ Z tanℑ = 0

$\frac{\sin ∀\cos ℑ \; Z \sin ℑ \cos ∀}{\cos ∀ \cos ℑ}$ X 0

$\frac{\sin(∀ Z ℑ)}{\cos ∀ \cos ℑ}$ X 0 sin(∀ Z ℑ) = 0 ∀ Z ℑ = *n*⇔ where *n* ***I***

∀ = *n*⇔+ ℑ, where *n* ***I***.

Illustration 20

Question: **Solve the equations:**

(i) **tan 3¯ Z J$\sqrt{3}$** **(ii) cot¯ Z J1**

Solution: (i) tan 3∀ X Z$\sqrt{3}$ = – tan$\frac{⇔}{3}$ X tan Z$\frac{⇔}{3}$

3∀ X *n*⇔ Γ Z$\frac{⇔}{3}$, *n* *I* ∀ X $\frac{n⇔}{3}$ Z $\frac{⇔}{9}$, *n* *I*

(ii) cot ∀ X Z1 tan ∀ X Z1 = tan Z$\frac{⇔}{4}$ ∀ X *n*⇔ Z $\frac{⇔}{4}$, *n* *I*

7.6 (a) GENERAL SOLUTION OF sin²¯ = sin²~, cos²¯ = cos²~

∀ = *n*⇔ ± ℑ, *n* *I*

(b) GENERAL SOLUTION OF tan²¯ = tan²~, sec²¯ = sec²~

∀ = *n*⇔ ± ℑ, *n* *I*, where ℑ | ƒ2*m* Γ 1Ą$\frac{⇔}{2}$, ¬*m* *I*

(c) GENERAL SOLUTION OF cot²¯ = cot²~, cosec²¯ = cosec²~

∀ = *n*⇔ ± ℑ, *n* *I*, where ℑ | *m*⇔ ¬*m* *I*

Illustration 21

Question: **Find the general solution of |cosec(¯ + ®/4)|** X $\frac{2}{\sqrt{3}}$.

Solution: Given equation is same as

cosec² ∀ Γ $\frac{⇔}{4}$ X $\frac{4}{3}$ X cosec² $\frac{⇔}{3}$

∀ Γ $\frac{⇔}{4}$ X *n*⇔ ± $\frac{⇔}{3}$ ∀ = *n*⇔ + $\frac{⇔}{12}$

or *n*⇔ Z $\frac{7⇔}{12}$, *n* *I*

8. EQUATION OF THE FORM : $a \cos\theta + b \sin\theta = c$

To solve the equation, we take

$a = r\cos\alpha \qquad r = \sqrt{a^2 + b^2}$

$b = r\sin\alpha \qquad$ and $\qquad \tan\alpha = \dfrac{b}{a}$

Substituting these values in the given equation, we have

$r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = c$

or $\quad r\cos(\theta - \alpha) = c$

or $\quad \cos(\theta - \alpha) \Rightarrow \dfrac{c}{r} = \dfrac{c}{\sqrt{a^2 + b^2}} = \cos\beta$ (say)

$\theta - \alpha = 2n\pi \pm \beta$

$\therefore \quad \theta = 2n\pi + \alpha \pm \beta$

Here α and β are constants which are dependent on a, b, c

Note: We can find the value of β provided $\left|\dfrac{c}{\sqrt{a^2 + b^2}}\right| \le 1$

[$\because$ $\cos\beta$ lies between -1 and 1]

or $\qquad |c| \le \sqrt{a^2 + b^2}$.

9. SOME USUAL TECHNIQUES TO SOLVE TRIGONOMETRIC EQUATIONS

- Check validity of the equation before solving, if possible.
- Squaring should be avoided as far as possible. If squaring is done, check for extraneous roots.
- Do not cancel terms containing ‘unknown’ on two sides of the equation. It may cause root loss.
- All solutions must come within the domain of the variable.
- The problems of trigonometric equations can be solved either by factorization method or by using the form $a\cos\theta + b\sin\theta$.
- Given a choice of converting equation of a given problem into either sine form or cosine form, then one should prefer the cosine form.

Illustration 22

Question: **Find general solution of $\sin^2\theta \sec\theta + \sqrt{3}\tan\theta = 0$.**

***Solution*:** $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0 \qquad \sin\theta(\sin\theta + \sqrt{3})\sec\theta = 0$

$\sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi, n \in I \qquad (\because \sin\theta \neq -\sqrt{3}, \sec\theta \neq 0)$

Illustration 23

Question: **Solve : $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2} \quad (0 \le x \le 2\pi)$.**

***Solution*:** The equation is $2\cos\frac{5x}{2}\cos\frac{x}{2} = 2\sin x\cos\frac{x}{2}$

$$2\cos\frac{x}{2}\left(\cos\frac{5x}{2} - \sin x\right) = 0$$

Either $\cos\frac{x}{2} = 0$ or $\cos\frac{5x}{2} = \sin x = \cos\left(\frac{\pi}{2} - x\right)$

$$\frac{x}{2} = 2n\pi \pm \frac{\pi}{2} \quad \text{or} \quad \frac{5x}{2} = 2k\pi \pm \left(\frac{\pi}{2} - x\right); \text{ where } n, k \in I$$

In $0 \le x \le 2\pi$ the solution set is $\left\{\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}\right\}$

PRACTICE PROBLEMS

PP1. Find the principal solutions of the following trigonometric equations:

(i) $\sin\theta = \frac{1}{2}$ (ii) $\tan\theta = -1$ (iii) $\cot\theta = \frac{1}{\sqrt{3}}$

(iv) $\operatorname{cosec}\theta = \sqrt{2}$ (v) $\sec\theta = -\frac{2}{\sqrt{3}}$ (vi) $\cos\theta = \frac{1}{\sqrt{2}}$

PP2. Find the general solution of the following trigonometric equations:

(i) $\sin 6\theta = 0$ (ii) $\cos k\theta = 0$ (iii) $\tan 7\theta = 0$

(iv) $\cot 5\theta = 0$

PP3. Find the general solution of the following trigonometric equations:

(i) $\sin 9\theta = \frac{\sqrt{3}}{2}$ (ii) $2\sin^2\theta - 5\sin\theta + 3 = 0$

PP4. Find the general solution of the following trigonometric equations:

(i) $\cos 5\theta = -\frac{1}{2}$ (ii) $\cos^2\theta - \frac{3}{\sqrt{2}}\cos\theta + 1 = 0$

PP5. Find the general solution of the following trigonometric equations:

(i) $\tan 2\theta = 1$ (ii) $\tan^2\theta - (\sqrt{3} - 1)\tan\theta - \sqrt{3} = 0$

PP6. Find the general solution of the equation $5\tan^4\theta - 1 = 4\tan^2\theta$.

PP7. Find the general solution of the following trigonometric equations:

(i) $\sqrt{2}\sec\theta + \tan\theta = 1$ (ii) $\sqrt{3}\sin\theta + \cos\theta = 3$

PP8. Find the length of an arc of a circle of radius 10 cm which subtends an angle of 45° at the centre.

PP9. Prove that: $\tan 720^\circ - \cos 270^\circ - \sin 150^\circ \cos 120^\circ = \frac{1}{4}$.

PP10. Prove that $\cos 105^\circ + \cos 15^\circ = \sin 75^\circ - \sin 15^\circ$.

PP11. If $A + B = \frac{\pi}{4}$, prove that $(1 + \tan A)(1 + \tan B) = 2$.

PP12. If $A + B + C = \pi$ prove that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$.

PP13. If $A + B + C = \pi$ prove that $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$.

PP14. Show that $\sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$.

PP15. Prove that $\cos\frac{\pi}{7}.\cos\frac{2\pi}{7}.\cos\frac{4\pi}{7} = -\frac{1}{8}$.

SOLVED SUBECTIVE EXAMPLES

***Example* 1:**

A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres ?

Solution :

The angle in radian measure $= \frac{25}{180}\pi = \frac{5}{36}\pi$

If r is the radius of the circle, using $l = r\theta$, we have

$$r = \frac{l}{\theta} = \frac{40}{\frac{5\pi}{36}} = \frac{288}{\pi} \text{ or } \frac{288 \times 7}{22} = 91.636 \text{ m.}$$

***Example* 2:**

Find the principal solution of the equation $\cot\left(\theta - \frac{\pi}{6}\right) = \sqrt{3}$.

***Solution*:**

The equation $\cot\left(\theta - \frac{\pi}{6}\right) = \sqrt{3}$ is satisfied

if $\quad \theta - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ or $-\frac{5\pi}{6}$, *etc*.

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{4\pi}{3}, -\frac{2\pi}{3}$, *etc*.

Out of these, the numerically smallest is $\theta = \pi/3$.

This is the principal solution. It is a particular solution also. Also the above values of θ taken individually are particular solutions.

Example 3:

Solve $\sin 2x + \cos x = 0$.

***Solution*:**

$\sin 2x + \cos x = 0$

$\Rightarrow \quad \cos x\{2\sin x + 1\} = 0$

$\Rightarrow \quad \cos x = 0$ or $\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{2}$ or $x = m\pi + (-1)^m \frac{7\pi}{6}$, $m, n \in I$

***Example* 4:**

Find the values of the other five trigonometric functions if $\cos\theta = -\frac{1}{2}$.

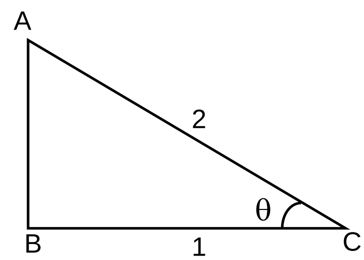
Solution :

$\because$ $\cos\theta$ is negative in IInd and IIIrd quadrant only

In IInd quadrant sine and cosecant is positive and other trigonometric ratios are negative.

Now construct a right angle triangle with base $BC = 1$ and hypotenuse $AC = 2$

$\Rightarrow$ perpendicular $AB = \sqrt{3}$



$\Rightarrow \quad \sin\theta = \frac{AB}{AC} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \operatorname{cosec}\theta = \frac{AC}{AB} = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad \tan\theta = -\frac{AB}{BC} = -\sqrt{3} \quad \Rightarrow \quad \cot\theta = -\frac{BC}{AB} = -\frac{1}{\sqrt{3}}$

and $\quad \sec\theta = -\frac{AC}{BC} = -2$

In IIIrd quadrant only tangent and cotangent are positive and rest are negative

$\Rightarrow \quad \sin\theta = -\frac{\sqrt{3}}{2}, \quad \tan\theta = \sqrt{3}, \quad \cot\theta = \frac{1}{\sqrt{3}}, \quad \sec\theta = -2$ and $\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}}$

***Example* 5:**

Prove that $\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1$.

Solution :

L.H.S. $= \sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°)$

$= -\sin 420° \cos 390° + \cos 660° \sin 330° \qquad [\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta]$

$= -\sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°) + \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°)$

$= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°) = -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1$

= R.H.S.

***Example* 6:**

Solve $\sin mx + \sin nx = 0$, $m \neq n$.

***Solution*:**

$\sin mx + \sin nx = 0 \quad \Rightarrow \quad 2\sin\frac{mx + nx}{2}\cos\frac{mx - nx}{2} = 0$

$\Rightarrow \quad \sin\left(\frac{m+n}{2}\right)x = 0 \quad$ or $\quad \cos\left(\frac{m-n}{2}\right)x = 0$

$\Rightarrow \quad \sin\left(\frac{m+n}{2}\right)x = 0 \quad \Rightarrow \quad \left(\frac{m+n}{2}\right)x = k\pi,\ k \in I \quad \Rightarrow \quad x = \frac{2k\pi}{m+n},\ k \in I$

$\Rightarrow \quad \cos\left(\frac{m-n}{2}\right)x = 0 \quad \Rightarrow \quad \left(\frac{\text{m}-\text{n}}{2}\right)\text{x} = (2\text{p}+1)\frac{\pi}{2},\ \text{p} \in \text{I} \quad \Rightarrow \quad \text{x} = \frac{(2\text{p}+1)\pi}{\text{m}-\text{n}},\ \text{p} \in \text{I}$

$\therefore$ The solution is $\text{x} = \frac{2\text{k}\pi}{\text{m}+\text{n}}, \frac{(2\text{p}+1)\pi}{\text{m}-\text{n}}, \text{k}, \text{p} \in \text{I}$

***Example* 7:**

Prove that (1 J sin¯ + cos¯)^2 = 2(1 J sin¯) (1 + cos¯).

Solution :

L.H.S. $= \{(1 - \sin\theta) + \cos\theta\}^2 = (1 - \sin\theta)^2 + \cos^2\theta + 2\cos\theta\,(1 - \sin\theta)$

$= (1 - \sin\theta)^2 + (1 - \sin^2\theta) + 2\cos\theta\,(1 - \sin\theta)$

$= (1 - \sin\theta)\,\{(1 - \sin\theta) + (1 + \sin\theta) + 2\cos\theta\}$

$= (1 - \sin\theta)(2 + 2\cos\theta) = 2(1 - \sin\theta)\,(1 + \cos\theta)$

***Example* 8:**

In a triangle *ABC*, in which *A* + *B* + *C* = ®.

Prove that (i) tan*A* + tan*B* + tan*C* = tan*A* tan*B* tan*C*

(ii) tan (B/C) tan (*C*/2) + tan (*C*/2) tan (*A*/2) + tan (*A*/2) tan (*B*/2) = 1.

Solution:

(i) $A + B = \pi - C = 180° - C$

$\tan(A + B) = \tan(180° - C) = -\tan C$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\tan A + \tan B = -\tan C\,(1 - \tan A \tan B)$

i.e., $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $(A/2 + B/2) = \pi/2 - C/2 = 90° - C/2$

$\tan(A/2 + B/2) = \tan(\pi/2 - C/2) = \cot C/2$

$\frac{\tan A/2 + \tan B/2}{1 - \tan(A/2)\tan(B/2)} = \frac{1}{\tan C/2}$

$\tan\frac{C}{2}\,(\tan A/2 + \tan B/2) = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$

$\therefore$ we get

$\tan(C/2)\tan(A/2) + \tan(B/2)\tan(C/2) + \tan(A/2)\tan(B/2) = 1.$

***Example* 9:**

Show that $\sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$

Solution :

L.H.S. $= \sin(90^\circ + 15^\circ) + \cos(90^\circ + 15^\circ) = \cos 15^\circ - \sin 15^\circ = \cos(45^\circ - 30^\circ) - \sin(45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ - \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} . \frac{1}{2} - \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$ = R.H.S.

***Example* 10:**

Solve: $\cos^2 x - \sin x - \frac{1}{4} = 0.$

Solution:

Replacing $\cos^2 x$ by $1 - \sin^2 x$, we get a quadratic in $\sin x$ in the form $4\sin^2 x + 4\sin x - 3 = 0$

i.e., $(2\sin x + 3)(2\sin x - 1) = 0 \Rightarrow \sin x \neq -\frac{3}{2}$ since $|\sin x| \le 1$

$\therefore \quad \sin x = \frac{1}{2}$. Principal solution is $x = \frac{\pi}{6}$

General solution is $x = n\pi + (-1)^n \frac{\pi}{6}$.

***Example* 11:**

Solve the equation $\sin 3x + \cos 2x = 0$.

Solution:

We have, $\sin 3x + \cos 2x = 0 \Rightarrow \cos 2x = -\sin 3x \Rightarrow \cos 2x = \cos\left(\frac{\pi}{2} + 3x\right)$

$\therefore \quad 2x = 2n\pi \pm \left(\frac{\pi}{2} + 3x\right), n \in I$

Taking positive sign $x = -2n\pi - \frac{\pi}{2}, n \in I$

Taking negative sign $x = \frac{2n\pi}{5} - \frac{\pi}{10}, n \in I$

***Example* 12:**

Show that $\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta) = 0$.

Solution :

L.H.S. $= \cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta)$

$= \cos\theta - \cos\theta + \cos\theta - \cos\theta = 0$ = R.H.S.

***Example* 13:**

Prove that $\tan 4\theta = \dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$

Solution :

L.H.S. = $\tan 4\theta$ = $\tan[2(2\theta)]$

$$= \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2.\dfrac{2\tan\theta}{1-\tan^2\theta}}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2} = \frac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2-4\tan^2\theta} = \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta} = \text{R.H.S}$$

***Example 14*:**

Solve the equation $\tan^3 x - 3\tan x = 0$.

Solution:

$\tan^3 x - 3\tan x = 0 \quad \Rightarrow \quad \tan x(\tan^2 x - 3) = 0$

$\Rightarrow \quad \tan x = 0$ or $\tan^2 x = 3 \quad \Rightarrow \quad x = n\pi$ or $x = m\pi \pm \dfrac{\pi}{3},\ m, n \in I$

***Example* 15:**

Solve the equations

(i) $\sec x - \tan x = \sqrt{3}$ **(ii)** $\sin x + \cos x = 1$

Solution:

(i) $\sec x - \tan x = \sqrt{3} \Rightarrow \dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x} = \sqrt{3} \Rightarrow \sqrt{3}\cos x + \sin x = 1$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$

we get $\dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x = \dfrac{1}{2} \Rightarrow \cos\dfrac{\pi}{6}\cos x + \sin\dfrac{\pi}{6}\sin x = \dfrac{1}{2} \Rightarrow \cos\left(x - \dfrac{\pi}{6}\right) = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow \quad x - \dfrac{\pi}{6} = 2n\pi \pm \dfrac{\pi}{3} \Rightarrow x = 2n\pi \pm \dfrac{\pi}{3} + \dfrac{\pi}{6},\ n \in I$

Taking positive sign: $x = 2n\pi + \dfrac{\pi}{2},\ n \in I$, which is not possible as sec x and tan x are not defined for these values of x.

Taking negative sign: $x = 2n\pi - \dfrac{\pi}{3} + \dfrac{\pi}{6} = 2n\pi - \dfrac{\pi}{6},\ n \in I$

(ii) $\sin x + \cos x = 1 \Rightarrow \dfrac{1}{\sqrt{2}}\sin x + \dfrac{1}{\sqrt{2}}\cos x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\dfrac{\pi}{4}\sin x + \cos\dfrac{\pi}{4}\cos x = \dfrac{1}{\sqrt{2}} \Rightarrow \cos\left(x - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4} \Rightarrow x - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4},\ n \in I$

$\therefore \quad x = 2n\pi \pm \dfrac{\pi}{4} + \dfrac{\pi}{4},\ n \in I$

EXERCISE–I

1. Find the values of the other five trigonometric functions for $\cot\theta = \frac{12}{5}, \theta$ lies in third quadrant.

2. Solve: (i) $\sin 7\theta = \sin 3\theta + \sin\theta$.

(ii) $\cos 3x + \cos x - \cos 2x = 0$

(iii) $\sin x + \sin 3x + \sin 5x = 0$

3. Show that: (i) $\frac{1+\sin A}{1+\cos A} \times \frac{1+\sec A}{1+\text{cosec}A} = \tan A$.

(ii) $\frac{\sin^4 A - \cos^4 A + \cos^2 A}{2(1-\cos A)} = \cos^2 \frac{A}{2}$.

(iii) $\tan 20° \tan 40° \tan 80° = \sqrt{3}$.

4. Prove that: (i) $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$.

(ii) $\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A$ J $\sin^2 B$.

5. Find the general values of θ satisfying the equation

(a) $\sin\theta = \frac{-1}{\sqrt{2}}$ (b) $\cos\theta = \frac{-1}{2}$ (c) $\tan\theta = -\sqrt{3}$

6. Find the general value of θ satisfying the two equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ simultaneously.

7. Prove that: (i) $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$.

(ii) $\frac{\cot\theta}{(\text{cosec}\theta + 1)} + \frac{(\text{cosec}\theta + 1)}{\cot\theta} = 2\sec\theta$.

8. If the equation $p\sin\theta + \cos 2\theta = 2p$ J 7 possesses a solution, then find the value(s) of p.

9. Simplify: $\frac{\sin\left(\frac{3\pi}{2} - \theta\right)\cos\left(\frac{\pi}{2} + \theta\right)}{\tan\left(\frac{\pi}{2} + \theta\right)} - \frac{\sin\left(\frac{3\pi}{2} - \theta\right)}{\sec(\pi + \theta)}$.

10. If $A + B + C = \pi$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.

11. Prove that $\text{cosec}^4\theta(1 - \cos^4\theta) = 1 + 2\cot^2\theta$.

12. Solve: $\sin x = \tan x$.

13. Prove that $\sin(n+1)x \, . \sin(n+2)x + \cos(n+1)x \, . \cos(n+2)x = \cos x$

14. Solve the equation $\tan 6\theta - \tan 4\theta = 0$.

15. Prove that $\dfrac{1 + \sin 2\theta - \cos 2\theta}{1 + \sin 2\theta + \cos 2\theta} = \tan\theta$.

EXERCISE – II

1. Find the value of $\cos\frac{\pi}{8}.\cos\frac{3\pi}{8}.\cos\frac{5\pi}{8}.\cos\frac{7\pi}{8}$.

2. Solve for x : $3^{\sin 2x+2\cos^2 x}+3^{1-\sin 2x+2\sin^2 x}=28$.

3. Prove that:: tan360° – cos270° – sin30° cos120°= $\sin^2 30°$

4. Show that the equation : $e^{\sin x}$ J $e^{-\sin x}-4=0$ has no real solution.

5. Prove that : $\frac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°}=\tan 54°$.

6. Find the values of x, $0\le x\le\frac{\pi}{2}$, which satisfying the equation $81^{\sin^2 x}+81^{\cos^2 x}=30$.

7. Prove that : $\cos^2 45°-\sin^2 15°=\frac{\sqrt{3}}{4}$.

8. Solve the equations:

(i) $\sin x+\cos x=\sqrt{2}$

(ii) $\sqrt{3}\cos x-\sin x=1$

9. Solve: $\sqrt{3}\cos\theta+\sin\theta=1, -2\pi<\theta<2\pi$.

10. If α and β be two distinct real numbers satisfying the equation $a\cos x+b\sin x=c$, then prove that

(i) $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$

(ii) $\tan(\alpha+\beta)=\frac{2ab}{a^2-b^2}$

11. If $A+B+C=0$, show that sin$A$ + sin$B$ + sin$C$ = J $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

12. If $A+B+C=\pi$, prove that

(i) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

13. If $A+B+C=\pi$, prove that $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$.

14. Solve the equation $\tan 2x=-\cot\left(x+\frac{\pi}{6}\right)$.

15. Solve $1+\sin^3 x+\cos^3 x=\frac{3}{2}\sin 2x$.

ANSWERS

ANSWERS TO PRACTICE PROBLEMS

PP1. (i) $\frac{\pi}{6}, \frac{5\pi}{6}$ (ii) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (iii) $\frac{\pi}{3}, \frac{4\pi}{3}$

(iv) $\frac{\pi}{4}, \frac{3\pi}{4}$ (v) $\frac{5\pi}{6}, \frac{7\pi}{6}$ (vi) $\frac{\pi}{4}, \frac{7\pi}{4}$

PP2 . (i) $\theta = \frac{n\pi}{6}, n \in I$ (ii) $\theta = (2n+1)\frac{\pi}{2k}, n \in I$ (iii) $\theta = \frac{n\pi}{7}, n \in I$

(iv) $\theta = (2n+1)\frac{\pi}{10}, n \in I$

PP3. (i) $\theta = \frac{n\pi}{9} + (-1)^n \frac{\pi}{27}, n \in I$ (ii) $\theta = 2n\pi + \frac{\pi}{2}, n \in I$

PP4. (i) $\theta = \frac{2n\pi}{5} \pm \frac{2\pi}{15}, n \in I$ (ii) $\theta = 2n\pi \pm \frac{\pi}{4}, n \in I$

PP5. (i) $\theta = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$ (ii) $\theta = n\pi + \frac{\pi}{3}$ or $\theta = m\pi + \frac{3\pi}{4}, m, n \in I$

PP6. $\theta = n\pi \pm \frac{\pi}{4}, n \in I$

PP7. (i) $\theta = 2n\pi - \frac{\pi}{4}, n \in I$ (ii) no solution

PP8. $\frac{5\pi}{2}$ cm

EXERCISE–I

1. $\sin\theta = -\frac{5}{13}, \cos\theta = -\frac{12}{13}, \tan\theta = \frac{5}{12}, \text{cosec}\,\theta = -\frac{13}{12},\ \sec\theta = -\frac{13}{12}.$

2. (i) $\frac{n\pi}{3}, \frac{m\pi}{2} \pm \frac{\pi}{12}, m, n \in I$

(ii) $x = (2m+1)\frac{\pi}{4}$, or $2n\pi \pm \frac{\pi}{3}$, $m,\ n \in I$

(iii) $x = \frac{m\pi}{3}$ or $n\pi \pm \frac{\pi}{3}$, $m, n \in I$

5. (a) $\theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right),\ n \in I$

(b) $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right),\ n \in I$

(c) $\theta = n\pi - \frac{\pi}{3},\ n \in I$

6. $2n\pi + \frac{7\pi}{6},\ n \in I$

8. $2 \le p \le 6$

9. –1

12. $x = n\pi,\ n \in I$

14. $\theta = \frac{n\pi}{2}$**,** $n \in I$

EXERCISE – II

1. $\frac{1}{8}$

2. $x = (2n+1)\frac{\pi}{2},\ x = m\pi - \frac{\pi}{4}; n, m \in I$

6. $\frac{\pi}{6}, \frac{\pi}{3}$

8. (i) $x = 2k\pi + \frac{\pi}{4};\ k \in I$ (ii) $x = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}, n \in I$

9. $-\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}$

14. $x = n\pi + \frac{2\pi}{3}, n \in I$

15. $x = 2n\pi + \frac{\pi}{4} \pm \frac{3\pi}{4}$